

Fixed Point Theorems for Multimaps Having KKM Property on G -convex
Uniform Spaces

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Abstract. We establish a new fixed point theorem for multimaps having KKM property on G -convex uniform spaces, not necessarily on locally G -convex uniform spaces. This result is parallel to our previous one for topological vector spaces in *Nonlinear Analysis Forum* 5 (2000), 51-59.

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1. Introduction

Many problems in nonlinear analysis can be formulated as the fixed point problems for multimaps on topological spaces without linear structure. In this direction, there have appeared a few kinds of abstract convexity notions. For example, the hyperconvex metric spaces due to Aronszajn and Panitchpakdi [1], the convex spaces due to Lassonde [8], the c -spaces due to Horvath [5] which are called the H -spaces by Bardaro and Ceppitelli [13]. These different kinds of concepts can be unified as G -convex spaces introduced by Park and Kim [9]. Since then, many results about fixed point theory and KKM theory on topological vector spaces were extended to G -convex spaces, cf. [10, 12, 15, 16, 17]. However, the underlying spaces for their fixed point results are locally G -convex uniform spaces. This paper intends to establish a fixed point theorem for multimaps having KKM property on G -convex uniform spaces, not necessarily on locally G -convex uniform spaces. This result is parallel to that for topological vector spaces in [6] which in turn revises and extends the Kim's fixed point theorem for lower semicontinuous multimaps in Hausdorff topological vector spaces as well as that of Park in [11].

We now recall some basic definitions and facts. Throughout this paper, $\langle Y \rangle$ denotes the class of all nonempty finite subsets of a nonempty set Y . The notation $T : X \multimap Y$ stands for a multimap from a set X into $2^Y \setminus \{\emptyset\}$. For a multimap $T : X \rightarrow 2^Y$, the following notations are used:

- (a) $T(A) = \bigcup_{x \in A} T(x)$ for $A \subseteq X$;
- (b) $T^-(y) = \{x \in X : y \in T(x)\}$ for $y \in Y$;
- (c) $T^-(B) = \{x \in X : T(x) \cap B \neq \emptyset\}$ for $B \subseteq Y$.
- (d) $T^c(x) = Y \setminus T(x)$ for $x \in X$.

All topological spaces are supposed to be Hausdorff. Let X and Y be two topological spaces. A multimap $T : X \rightarrow 2^Y$ is said to be

- (a) upper semicontinuous (u.s.c.) if $T^-(B)$ is closed in X for each closed subset B of Y ;
- (b) lower semicontinuous (l.s.c.) if $T^-(B)$ is open in X for each open subset B

of Y ;

(c) compact if $T(X)$ is contained in a compact subset of Y ;

(d) closed if its graph $\text{Gr}(T) = \{(x, y) : y \in T(x), x \in X\}$ is a closed subset of $X \times Y$.

It is well-known that if Y is a regular space and T is a closed-valued u.s.c. multimap, then T is closed. The converse is true if Y is compact, cf.[8 Lemma 1].

$\bar{T} : X \rightarrow 2^Y$ is defined to be $\bar{T}(x) = \overline{T(x)}$ for each $x \in X$.

The unit coordinate vectors in \mathbb{R}^{n+1} are denoted by $\mathbf{e}_0, \dots, \mathbf{e}_n$, and Δ_n stands for the standard n -simplex of \mathbb{R}^{n+1} , that is,

$$\Delta_n = \left\{ (\lambda_0, \dots, \lambda_n) : \lambda_i \geq 0 \text{ for all } i \text{ and } \sum_{i=0}^n \lambda_i = 1 \right\}.$$

Later on, if $J \in \langle \{0, 1, \dots, n\} \rangle$, then $\text{co}\{\mathbf{e}_i : i \in J\}$ will be denoted by Δ_J .

Park and Kim [9] unified many different kinds of abstract convexity in the following way.

Definition 1.1. A generalized convex space or a G -convex space $(X, D; \Gamma)$ consists of a topological space X , a nonempty subset D of X and a map $\Gamma : \langle D \rangle \rightarrow X$ such that for each $A = \{a_0, \dots, a_n\} \in \langle D \rangle$ with $|A| = n + 1$, there exists a continuous function $\varphi_A : \Delta_n \rightarrow \Gamma(A)$ such that if $0 \leq i_0 < i_1 < \dots < i_k \leq n$, then $\varphi_A(\text{co}\{\mathbf{e}_{i_0}, \dots, \mathbf{e}_{i_k}\}) \subseteq \Gamma(\{a_{i_0}, \dots, a_{i_k}\})$. When $D = X$, $(X, D; \Gamma)$ is abbreviated to $(X; \Gamma)$. In this paper, we assume that a G -convex space $(X, D; \Gamma)$ always satisfies the extra condition: $x \in \Gamma(\{x\})$ for any $x \in X$.

A subset K of a G -convex space (X, Γ) is said to be Γ -convex if for any $A \in \langle K \rangle$, $\Gamma(A) \subseteq K$.

For convenience, we also express $\Gamma(A)$ by Γ_A .

A uniformity for a set X is a nonempty family \mathcal{U} of subsets of $X \times X$ such that

- (a) each member of \mathcal{U} contains the diagonal Δ ;
- (b) if $U \in \mathcal{U}$, then $U^{-1} \in \mathcal{U}$;
- (c) if $U \in \mathcal{U}$, then $V \circ V \subseteq U$ for some V in \mathcal{U} ;
- (d) if U and V are members of \mathcal{U} , then $U \cap V \in \mathcal{U}$; and
- (e) if $U \in \mathcal{U}$ and $U \subseteq V \subseteq X \times X$, then $V \in \mathcal{U}$.

Every member V in \mathcal{U} is called an entourage. An entourage V is said to symmetric if $(x, y) \in V$ whenever $(y, x) \in V$.

If (X, \mathcal{U}) is a uniform space the topology \mathcal{T} induced by \mathcal{U} is the family of all subsets W of X such that for each x in W there is U in \mathcal{U} such that $U[x] \subseteq W$, where $U[x]$ is defined as $\{y \in X : (x, y) \in U\}$. For details of uniform spaces we refer to [7].

Definition 1.2. A G -convex uniform space $(E; \Gamma, \mathcal{U})$ is a G -convex space so that its topology is induced by a uniformity \mathcal{U} .

A G -convex uniform space $(E; \Gamma, \mathcal{U})$ is said to be a locally G -convex uniform space if it satisfies the following conditions:

- (a) $\Gamma_{\{x\}} = \{x\}$ for any $x \in E$;
- (b) The uniformity \mathcal{U} has a base \mathcal{B} consisting of open symmetric entourages such that for each $V \in \mathcal{B}$, $V[K] := \{y \in X : (x, y) \in V \text{ for some } x \in K\}$ is Γ -convex whenever K is Γ -convex.

By the definition of G -convex uniform space $(E; \Gamma, \mathcal{U})$, it is easy to check that $A \subseteq \Gamma_A$ for any $A \in \langle E \rangle$.

For a G -convex space $(E; \Gamma)$, a multimap $F : E \multimap E$ is called a KKM map if $\Gamma_A \subseteq F(A)$ for each $A \in \langle E \rangle$. The following result is well-known.

Theorem 1.3. (The KKM Principle). Let D be the set of vertices of Δ_n and $F : D \multimap \Delta_n$ be a KKM map (that is $co(A) \subseteq F(A)$ for each $A \in \langle D \rangle$).

Then $\bigcap_{a \in D} \overline{F(a)} \neq \emptyset$.

The following known results in the literature will be quoted in the sequel.

Lemma 1.4 ([14]). Suppose X is a compact topological space and $(Y; \Gamma)$ a G -convex space. If $T : X \multimap Y$ satisfies

- (a) $T(x)$ is Γ -convex for each $x \in X$;
- (b) $T^-(y)$ is open in X for each $y \in Y$,

then T has a continuous selection.

Lemma 1.5.([16]) Let $(X; \Gamma)$ be a compact locally G -convex uniform space. If $p : X \rightarrow \Delta_n$ is continuous and $Q : \Delta_n \multimap X$ is u.s.c. with compact and Γ -convex values, then $p \circ Q : \Delta_n \multimap \Delta_n$ has a fixed point.

Lemma 1.6.([5]) Let X, Y be two topological spaces. If $T : X \multimap Y$ is l.s.c. and $g : X \rightarrow Y$ is continuous and $V : Y \multimap Y$ has open graph, then $x \rightarrow T(x) \cap V(g(x))$ is l.s.c.

2. The Main Results

In this section, we shall establish a new fixed point theorem for multimaps having KKM property in G -convex uniform spaces. To begin with, recall that

Definition 2.1. Suppose (X, Γ) is a G -convex space and Y a topological space. If $F, T : X \multimap Y$ are two multimaps satisfying that

$$T(\Gamma_A) \subseteq F(A)$$

for each $A \in \langle X \rangle$, then F is called a KKM map with respect to T . If the multimap T satisfies the requirement that for any KKM map F with respect to T the family $\{\overline{F(x)} : x \in X\}$ has the finite intersection property, then T is said to have the KKM property. The collection of all multimaps from X to Y that have the KKM property is denoted by $\text{KKM}(X, Y)$.

In what follows, $\Gamma(V[y])$ is defined as $\Gamma(V[y]) = \bigcup_{A \in \langle V[y] \rangle} \Gamma_A$

Lemma 2.2. Let X be a nonempty Γ -convex subset of a G -convex uniform space $(E; \Gamma, \mathcal{U})$ which has a uniformity \mathcal{U} with a base \mathcal{B} of open symmetric entourages. Suppose that for any $V \in \mathcal{B}$, $x \in \Gamma(V[y])$ if and only if $y \in \Gamma(V[x])$. If $T : X \multimap X$ is compact and has the KKM property, then for any $V \in \mathcal{B}$, there is $x_V \in X$ such that $\Gamma(V[x_V]) \cap T(x_V) \neq \emptyset$.

PROOF. On the contrary, assume there is $V \in \mathcal{B}$ such that for any $x \in X$, $\Gamma(V[x]) \cap T(x) = \emptyset$. Let $K = \overline{T(X)}$. Then K is compact in X . Define $F : X \rightarrow 2^X$ by $F(x) = K \setminus V[x]$. Clearly, for each $x \in X$, $F(x)$ is closed. And using the fact that $x \in \Gamma_{\{x\}}$ it is easy to check that $F(x)$ is nonempty. We will show that F is a KKM map with respect to T . If not, there is $A = \{x_1, \dots, x_n\} \in \langle X \rangle$ such that $T(\Gamma_A) \not\subseteq F(A)$. Choose $z \in T(\Gamma_A)$ such that $z \notin F(A)$. Then there is $y \in \Gamma_A$ such that $z \in T(y)$ but $z \notin F(A)$. Since

$$\begin{aligned} z \notin F(A) &= \bigcup_{i=1}^n F(x_i) \\ &= \bigcup_{i=1}^n (K \setminus V[x_i]) \\ &= K \setminus \bigcap_{i=1}^n V[x_i], \end{aligned}$$

we conclude that $z \in V[x_i]$ for all $i = 1, \dots, n$, that is, $(x_i, z) \in V$ for all $i = 1, \dots, n$. By symmetry, $(z, x_i) \in V$ for all $i = 1, \dots, n$. So, $x_i \in V[z]$ for all $i = 1, \dots, n$. Consequently, $A \subseteq V[z]$, and hence $\Gamma_A \subseteq \Gamma(V[z])$. In particular, $y \in \Gamma(V[z])$. By hypothesis, $z \in \Gamma(V[y])$. Therefore, we reach the conclusion that $z \in T(y) \cap \Gamma(V[y])$, a contradiction. So, F is a KKM map with respect to T . Since T has the KKM property, $\{F(x) : x \in X\}$ has the finite intersection property. Now, noting that for each $x \in X$, $F(x) \subseteq K$ and K is compact, we infer that $\bigcap_{x \in X} F(x) \neq \emptyset$. Choose $\xi \in \bigcap_{x \in X} F(x)$. Then $\xi \in F(\xi) = K \setminus V[\xi]$, which implies that $\xi \notin V[\xi]$, a contradiction. This completes the proof. \square

Corollary 2.3. Suppose $(E; \Gamma, \mathcal{U})$ is a locally G -convex uniform space with a base \mathcal{B} of symmetric open entourages and X is a Γ -convex subset of E . If $T : X \multimap X$ is compact and has the KKM property, then for any $V \in \mathcal{B}$, there is $x_V \in X$ such that $V[x_V] \cap T(x_V) \neq \emptyset$.

PROOF. Since $\Gamma(V[x]) = V[x]$ in a locally G -convex uniform space, the result follows immediately. \square

Theorem 2.4. Suppose X is a nonempty compact Γ -convex subset of a G -convex uniform space $(E; \Gamma, \mathcal{U})$ with a base \mathcal{B} of symmetric open entourages such that for any $V \in \mathcal{B}$, $x \in \Gamma(V[y])$ if and only if $y \in \Gamma(V[x])$. If $T \in \text{KKM}(X, X)$ is closed-valued and satisfies the following condition (*):

(*) If $y \in X$ satisfies that $y \notin U[T(y)]$ for some open entourage U , then $y \notin \text{cl}\{x \in X : x \in \Gamma(V[T(x)])\}$ for some $V \in \mathcal{B}$, then T has a fixed point.

PROOF. For each $U \in \mathcal{B}$, put $F_U = \{x \in X : x \in \Gamma(U[T(x)])\}$ and $G_U = \{x \in X : x \in U[T(x)]\}$. By Lemma 2.2, there is $x_U \in X$ such that $\Gamma(U[x_U]) \cap T(x_U) \neq \emptyset$. Choose $y \in \Gamma(U[x_U]) \cap T(x_U)$. It follows from the assumption that $x_U \in \Gamma(U[y])$, and hence $x_U \in \Gamma(U[T(x_U)])$. Consequently, for each $U \in \mathcal{B}$, $F_U \neq \emptyset$. Since $\{\overline{F_U} : U \in \mathcal{B}\}$ has the finite intersection property and since X is compact, we conclude that $\bigcap_{U \in \mathcal{B}} \overline{F_U} \neq \emptyset$. Also, it is obvious that $\bigcap_{U \in \mathcal{B}} G_U \subseteq \bigcap_{U \in \mathcal{B}} \overline{F_U}$ by noting that $G_U \subseteq \overline{F_U}$ for each $U \in \mathcal{B}$. For the reverse inclusion, if there is $y \in \bigcap_{U \in \mathcal{B}} \overline{F_U}$ such that $y \notin \bigcap_{U \in \mathcal{B}} G_U$, then $y \notin U[T(y)]$ for some $U \in \mathcal{B}$, hence by (*) we can choose $V \in \mathcal{B}$ such that $y \notin \overline{F_V}$, a contradiction. Therefore, $\bigcap_{U \in \mathcal{B}} G_U = \bigcap_{U \in \mathcal{B}} \overline{F_U} \neq \emptyset$. Letting $\xi \in \bigcap_{U \in \mathcal{B}} G_U$, we see

$$\xi \in \bigcap_{U \in \mathcal{B}} U[T(\xi)] = T(\xi),$$

where the equality holds is due to $T(\xi)$ is closed. This completes the proof. \square

When X is a topological vector space, this is our previous result [6, Theorem 2.2].

In a locally G -convex uniform space, we have $\Gamma(V[x]) = V[x]$, and so $y \in V[x]$ if and only if $x \in V[y]$ for any symmetric open entourage V .

Proposition 2.5. Suppose X is a nonempty compact Γ -convex subset of a locally G -convex uniform space $(E; \Gamma, \mathcal{U})$ and $T : X \multimap X$ is closed and has Γ -convex values, then the condition (*) holds:

(*) If $y \in X$ satisfies that $y \notin U[T(y)]$ for some open entourage U , then $y \notin \text{cl}\{x \in X : x \in \Gamma(V[T(x)])\}$ for some $V \in \mathcal{B}$

PROOF. Choose $V \in \mathcal{B}$ such that $V \subseteq \bar{V} \subseteq U$. Then the set $A := \{x \in X : x \in \bar{V}[T(x)]\}$ is closed. Indeed, for any $x \in \bar{A}$, choose a net $\{x_\alpha\}$ in A such that $x_\alpha \rightarrow x$. Since $x_\alpha \in A$ we can choose $z_\alpha \in T(x_\alpha)$ such that $(x_\alpha, z_\alpha) \in \bar{V}$. By the compactness of X , we may assume that $z_\alpha \rightarrow z$ for some $z \in X$. So $(x_\alpha, z_\alpha) \rightarrow (x, z)$, and hence $(x, z) \in \bar{V}$ by noting that $\{(x_\alpha, z_\alpha)\} \subset \bar{V}$ and \bar{V} is closed. Meanwhile, since T is closed, we have that $z \in T(x)$, so $x \in \bar{V}[T(x)] \cap X$, which shows that A is closed. Now, if $y \notin U[T(y)] \cap X$, then $y \notin \bar{V}[T(y)] \cap X$, and so $y \notin A = \bar{A}$. Hence $y \notin \text{cl}\{x \in X : x \in V[T(x)]\}$. Moreover, since $V[T(x)]$ is Γ -convex, $\Gamma(V[T(x)]) = V[T(x)]$. Therefore,

$$y \notin \text{cl}\{x \in X : x \in V[T(x)]\} = \text{cl}\{x \in X : x \in \Gamma(V[T(x)])\}.$$

□

Corollary 2.6. Suppose X is a nonempty compact Γ -convex subset of a locally G -convex uniform space $(E; \Gamma, \mathcal{U})$. If $T \in \text{KKM}(X, X)$ is closed and has Γ -convex values, then T has a fixed point.

PROOF. This follows from Theorem 2.4 and Proposition 2.5. □

Here we like to mention that the condition that T has Γ -convex values can be dropped by means of Corollary 2.3 and the technique in [3, Theorem 3.2].

3. The KKM Class

In this section, we study the KKM class more.

Proposition 3.1. If $f : X \rightarrow Y$ is a continuous function from a G -convex space (X, Γ) to a topological space Y , then $f \in \text{KKM}(X, Y)$.

PROOF. Let $F : X \multimap Y$ be any closed-valued KKM map with respect to f and $A = \{a_0, \dots, a_n\}$ be any member in $\langle X \rangle$. Since X is a G -convex space, there is a continuous function $\phi_A : \Delta_n \rightarrow \Gamma_A$ such that, for any $0 \leq i_0 < \dots < i_k \leq n$,

$$\phi_A(\text{co}(e_{i_0}, \dots, e_{i_k})) \subseteq \Gamma_{\{a_{i_0}, \dots, a_{i_k}\}} \cap \phi_A(\Delta_n),$$

so

$$\text{co}(e_{i_0}, \dots, e_{i_k}) \subseteq \phi_A^- \left(\Gamma_{\{a_{i_0}, \dots, a_{i_k}\}} \cap \phi_A(\Delta_n) \right). \quad (1)$$

Also, since F is a KKM map with respect to f , we have

$$f \left(\Gamma_{\{a_{i_0}, \dots, a_{i_k}\}} \right) \subseteq F(\{a_{i_0}, \dots, a_{i_k}\}),$$

that is,

$$\Gamma_{\{a_{i_0}, \dots, a_{i_k}\}} \subseteq f^- (F(\{a_{i_0}, \dots, a_{i_k}\})) \quad (2)$$

It follows from (1) and (2) that

$$\text{co}(e_{i_0}, \dots, e_{i_k}) \subseteq \phi_A^- (f^- (F(\{a_{i_0}, \dots, a_{i_k}\})) \cap \phi_A(\Delta_n)).$$

Noting that $e_i \dashv \phi_A^- (f^- (F(a_i)) \cap \phi_A(\Delta_n))$ is a KKM map, we infer from the KKM principle that

$$\bigcap_{i=0}^n \phi_A^- (f^- (F(a_i)) \cap \phi_A(\Delta_n)) \neq \emptyset,$$

which implies that $\bigcap_{i=0}^n F(a_i) \neq \emptyset$. Hence $f \in \text{KKM}(X, Y)$. \square

Lemma 3.2. Let X be a compact topological space and $(Y; \mathcal{U}, \Gamma)$ a locally G -convex uniform space. If $T : X \dashv Y$ is a l.s.c. multimap with Γ -convex values, then for any symmetric entourage V of the uniformity of Y , there exists a continuous function $f : X \rightarrow Y$ such that, for any $x \in X$, $T(x) \cap V[f(x)] \neq \emptyset$.

PROOF. We may assume that V is open in $Y \times Y$. For each $x \in X$, taking $y \in T(x)$, we see that $y \in T(x) \cap V[y]$. Define $F : X \dashv Y$ by $F(x) = \{y \in Y : T(x) \cap V[y] \neq \emptyset\}$. Since $V[T(x)] = \{y \in Y : (z, y) \in V \text{ for some } z \in T(x)\}$, we have $F(x) = V[T(x)]$ for each $x \in X$. Due to $T(x)$ is Γ -convex, we infer that $V[T(x)]$ is Γ -convex, so $F(x)$ is Γ -convex. Also, for any $y \in Y$, since

$$\begin{aligned} F^-(y) &= \{x \in X : y \in F(x)\} \\ &= \{x \in X : T(x) \cap V[y] \neq \emptyset\} \\ &= T^-(V[y]) \end{aligned}$$

and since $V[y]$ is open in Y , it follows from the lower semi-continuity of T that $F^-(y)$ is open in X . By Lemma 1.4, F has a continuous selection $f : X \rightarrow Y$, so $f(x) \in F(x)$, which implies that $T(x) \cap V[f(x)] \neq \emptyset$. \square

Definition 3.3. Let (Y, Γ) be a G -convex space. We call it a locally G -convex metric space if (Y, d) is a metric space such that for any $\varepsilon > 0$,

$$\{y \in Y : d(y, E) < \varepsilon\}$$

is a Γ -convex set whenever $E \subseteq Y$ is a Γ -convex set, and all open balls are Γ -convex.

Proposition 3.4. Let X be a compact topological space and (Y, Γ) a complete locally G -convex metric space. If $T : X \multimap Y$ is a l.s.c. multimap with closed Γ -convex values, then T has a continuous selection.

PROOF. For each $n \in \mathbb{N} \cup \{0\}$, let $V_n[y] = \{z \in Y : d(y, z) < \frac{1}{2^n}\}$. By Lemma 3.2, there exists a continuous function $f_0 : X \rightarrow Y$ such that $T(x) \cap V_0[f_0(x)] \neq \emptyset$. Define $T_1 : X \multimap Y$ by $T_1(x) = T(x) \cap V_0[f_0(x)]$. By Lemma 1.6, T_1 is l.s.c. Obviously, for each $x \in X$, $T_1(x)$ is Γ -convex. So, it follows from Lemma 3.2 that there exists a continuous function $f_1 : X \rightarrow Y$ such that $T_1(x) \cap V_1[f_1(x)] \neq \emptyset$. Continuing in this manner, we obtain a sequence of multimaps $T_n : X \multimap Y$ and a sequence of continuous functions $f_n : X \rightarrow Y$ such that for each $n \in \mathbb{N} \cup \{0\}$ and for each $x \in X$, $T_n(x) \cap V_n[f_n(x)] \neq \emptyset$. Since $d(f_{n+1}(x), f_n(x)) \leq \frac{1}{2^{n+1}} + \frac{1}{2^n}$ for all $x \in X$, the sequence $\{f_n\}$ is uniformly Cauchy, its limit function f is continuous and it is clear that $f(x) \in T(x)$ for each $x \in X$. \square

Proposition 3.5. Let (X, Γ) be a compact locally G -convex metric space. If $T : X \multimap X$ is a l.s.c. multimap with closed Γ -convex values, then $T \in \text{KKM}(X, X)$.

PROOF. By Proposition 3.4, T has a continuous selection $f : X \rightarrow X$. Since every KKM map F with respect to T is also a KKM map with respect to f and since $f \in \text{KKM}(X, X)$, it follows immediately that $T \in \text{KKM}(X, X)$. \square

Corollary 3.6. Let X be a nonempty compact Γ -convex subset of a locally G -convex uniform space $(E; \Gamma, \mathcal{U})$. If $T : X \multimap X$ is a l.s.c. multimap with closed Γ -convex values, then T has a fixed point.

PROOF. By Proposition 3.4, T has a continuous selection t which has a fixed point by Park [12, Theorem 3]. Therefore T has a fixed point. \square

Proposition 3.7. Let (X, Γ) be a compact locally G -convex uniform space. If $T : X \multimap X$ is a u.s.c. closed-valued multimap with Γ -convex values, then $T \in \text{KKM}(X, X)$.

PROOF. If $T \notin \text{KKM}(X, X)$, then there exists a closed-valued KKM map $F : X \multimap X$ with respect to T and a nonempty finite subset $A = \{x_0, \dots, x_n\}$ of X such that $\bigcap_{i=0}^n F(x_i) = \emptyset$. Since (X, Γ) is a G -convex space, there exists a continuous function $\phi_A : \Delta_n \rightarrow X$ satisfying that

- (i) $\phi_A(\Delta_n) \subseteq \Gamma_A$, and
- (ii) $\phi_A(\Delta_J) \subseteq \Gamma_{\{x_i : i \in J\}}$, for any $J \in \langle \{0, 1, \dots, n\} \rangle$.

Moreover, since $X = \bigcup_{i=0}^n F^c(x_i)$, it has a partition of unity $\{\alpha_i\}_0^n$ subordinated to the open covering $\{F^c(x_i)\}_0^n$ of X . Define $f : X \rightarrow \Delta_n$ by $f(x) = \sum_{i=0}^n \alpha_i(x)e_i$. Obviously, f is continuous, and so $T \circ \phi_A : \Delta_n \multimap X$ is u.s.c. with compact and Γ -convex values. By Watson [16] $f \circ T \circ \phi_A : \Delta_n \multimap \Delta_n$ has a fixed point \hat{x} . Choose $\hat{y} \in T \circ \phi_A(\hat{x})$ such that $\hat{x} = f(\hat{y})$. Denote J the set of indices i such that $\alpha_i(\hat{y}) > 0$. Then $\hat{y} \in F^c(x_i)$ for all $i \in J$, $\hat{x} = f(\hat{y}) \in \Delta_J$, and $\phi_A(\hat{x}) \in \phi_A(\Delta_J) \subseteq \Gamma_{\{x_i : i \in J\}}$. So $\hat{y} \in T(\Gamma_{\{x_i : i \in J\}})$, and hence $T(\Gamma_{\{x_i : i \in J\}}) \not\subseteq \bigcup_{i \in J} F(x_i)$. This is a contradiction. Therefore, $T \in \text{KKM}(X, X)$. \square

Corollary 3.8. Suppose X is a nonempty compact Γ -convex subset of a locally G -convex uniform space $(E; \Gamma, \mathcal{U})$. If $T : X \multimap X$ is u.s.c. with closed and Γ -convex values, then T has a fixed point.

PROOF. Since T is u.s.c. and closed-valued, it is closed, and so the result follows from Proposition 3.7 and Corollary 2.6. \square

This corollary is due to Watson [16, Theorem 4.1].

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