Fixed Points of Multimaps On Admissible Almost Convex Sets In Topological Vector Spaces

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We investigate the fixed point problem for a multimap T having the strict KKM property on an admissible almost convex subset of a topological vector space E. As applications of our new fixed point theorems, some results related to equilibrium problem in game theory are also deduced.

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1. Introduction

For a nonempty set Y, 2^Y denotes the class of all subsets of Y and $\langle Y\rangle$ the class of all nonempty finite subsets of Y. A multimap $T : X \to 2^Y$ is a function from a set X into the power set 2^Y of Y. The notation $T : X \multimap Y$ stands for a multimap $T : X \to 2^Y$ having nonempty values.

Recently, in order to study the fixed point problem for multimaps on not necessarily convex sets, the first two authors together with H. C. Hsu [5] introduced the following concepts of generalized KKM mapping and the strict KKM property:

1.1. Definition. *Suppose X and Y are two nonempty subsets of a linear space E, and T, $S : X \multimap Y$. We say that S is a generalized KKM mapping with respect to T if for any $A = \{x_1, \cdots, x_n\} \in \langle X\rangle$, there is $B = \{y_1, \cdots, y_n\} \in \langle X\rangle$ satisfying*

(a) *$co(B) \subseteq X$, and*

(b) *$T(co\{y_i : i \in I\}) \subseteq \bigcup_{i\in I} S(x_i)$ for any nonempty subset I of $\{1, \cdots, n\}$.*
Moreover, if E is a topological vector space and if a multimap $T : X \multimap Y$ satisfies that for any generalized KKM mapping $S : X \multimap Y$ with respect to T, the family $\{\overline{S(x)} : x \in X\}$ has the finite intersection property, then T is said to have the strict KKM property. The class SKKM(X, Y) is defined to be the set $\{T : X \multimap Y : T$ has the strict KKM property$\}$.

A new fixed point theorem for $T \in \mathrm{SKKM}(X, X)$ with X an almost convex subset of a locally convex topological vector space E was established in [5], which contains Lassonde's fixed point theorem for Kakutani factorizable multimaps as a special case. On the other hand, Klee [6] introduced the concept of admissible topological vector spaces and obtained fixed point results on such spaces. Since the admissibility can be used to eliminate the local convexity of the space itself, subsequent researches on admissible subsets have been followed until now. See Hadžić [2], Weber [11] and Park [9, 10]. The main purpose of this paper is to investigate the fixed point problem for $T \in \mathrm{SKKM}(X, X)$ with X an admissible almost convex subset of a topological vector space E. In section 2, we show that any almost convex subset of a locally convex topological vector space is admissible, which is an improvement for the Nagumo's result that says any convex subset of a locally convex topological vector space is admissible. Section 3 is devoted to the establishment of some new fixed point theorems for multimaps having the strict KKM property on admissible almost convex subsets of topological vector spaces. As applications of our new fixed point theorems, some results related to equilibrium problem in game theory are also deduced in section 4.

We now recall some basic facts. For a multimap $T : X \to 2^Y$, $A \subseteq X$ and $B \subseteq Y$, the image of A under T is the set $T(A) = \bigcup_{x \in A} T(x)$; and the inverse image of B under T is $T^-(B) = \{x \in X : T(x) \cap B \neq \varnothing\}$.

All topological spaces are supposed to be Hausdorff. The closure of a subset X of a topological space is denoted by $\overline{X}$. Let X and Y be two topological spaces. A multimap $T : X \to 2^Y$ is said to be

(a) upper semicontinuous(u.s.c.) if $T^-(B)$ is closed in X for each closed subset B of Y;

(b) lower semicontinuous(l.s.c.) if $T^-(B)$ is open in X for each open subset B of Y;

(c) compact if $T(X)$ is contained in a compact subset of Y;

(d) closed if its graph $\mathrm{Gr}(T) = \{(x, y) : y \in T(x), x \in X\}$ is a closed subset of $X \times Y$.

Lemma 1.2 (Lassonde[7, Proposition 1]). *Let X, Y, Z and X_i, Y_i $(i = 1, 2)$ be topological spaces and $T : X \multimap Y$.*

(a) *If Y is regular and T is u.s.c. with closed values, then T is closed.*

(b) *If $T : X \multimap Y$ is u.s.c. with compact values and $S : X \multimap Y$ is closed, then $T \cap S : X \multimap Y$, defined by $(T \cap S)(x) = T(x) \cap S(x)$ for each $x \in X$, is u.s.c.; in particular, if S is compact and closed, then S is u.s.c.*

(c) *If T is u.s.c. and compact-valued, then $T(A)$ is compact for any compact subset A of X.*

(d) *If $T : X \multimap Y$ and $S : Y \multimap Z$ are u.s.c., then the composition $ST : X \multimap Z$, defined by $S(T(x)) = \bigcup_{y \in T(x)} S(y)$ for each $x \in X$, is u.s.c.*

(e) *If $T_i : X_i \multimap Y_i$ $(i = 1, 2)$, are u.s.c., then*

$$T_1 \times T_2 : X_1 \times X_2 \multimap Y_1 \times Y_2$$

defined by $(T_1 \times T_2)(x_1, x_2) = T_1(x_1) \times T_2(x_2)$ for $(x_1, x_2) \in X_1 \times X_2$, is u.s.c.

$\mathcal{C}(X, Y)$ denotes the class of all continuous (single-valued) functions from X to Y.

2. Admissible Almost Convex Sets

Before we proceed to deal with fixed points of multimaps on admissible almost convex sets, we give some basic facts about such sets.

Definition 2.1 (Himmelberg[3]). *A nonempty subset X of a topological vector space E is said to be almost convex if for any neighborhood V of 0 in E and for any $\{x_1, \cdots, x_n\} \in \langle X \rangle$ there is $\{y_1, \cdots, y_n\} \in \langle X \rangle$ such that $y_i - x_i \in V$ for each $i \in \{1, \cdots, n\}$ and $co\{y_1, \cdots, y_n\} \subseteq X$.*

The following proposition is well-known, cf.[4]. For the sake of completeness, we give a proof here.

Proposition 2.2. *If X is an almost convex subset of a topological vector space E, then $\overline{X} = \overline{co}X$. In particular, $\overline{X}$ is convex.*

Proof. Let $z \in \overline{co}X$ and U be any neighborhood of the origin 0 in E. Since every neighborhood of 0 contains a balanced neighborhood of 0 and since vector addition is continuous, there is a balanced neighborhood V of 0 such that $V + V \subseteq U$. Choose $x = \sum_{i=1}^n \lambda_i x_i \in coX$ such that $x \in z + V$ and choose balanced neighborhoods V_i, $i = 1, \cdots, n$, of 0 such that $V_1 + \cdots + V_n \subseteq V$. Let $W = \cap_{i=1}^n V_i$. Since X is almost convex, there is $\{y_1, \cdots, y_n\} \in \langle X \rangle$ such that $co\{y_1, \cdots, y_n\} \subseteq X$ and $y_i - x_i \in W$ for all $i = 1, \cdots, n$. So $\sum_{i=1}^n \lambda_i y_i \in X$ and

$$\begin{aligned}\sum_{i=1}^n \lambda_i y_i - \sum_{i=1}^n \lambda_i x_i = \sum_{i=1}^n \lambda_i (y_i - x_i) &\in \sum_{i=1}^n \lambda_i W \\ &\subseteq \sum_{i=1}^n \lambda_i V_i \subseteq V.\end{aligned}$$

Consequently, $\sum_{i=1}^n \lambda_i y_i \in X$ and $\sum_{i=1}^n \lambda_i y_i \in x + V \subseteq z + V + V \subseteq z + U$. This shows that $\overline{co}X \subseteq \overline{X}$, and so $\overline{X} = \overline{co}X$. □

Definition 2.3 (Klee[6]). *A nonempty subset X of a topological vector space E is said to be admissible provided that, for every nonempty compact subset K of X and every neighborhood V of the origin 0 of E, there exists a continuous function $h : K \to X$ such that $x - h(x) \in V$ for all $x \in K$ and $h(K)$ is contained in a finite dimensional subspace L of E.*

Nagumo [8] proved that all convex subsets of a locally convex topological vector space are admissible. This result remains true for all almost convex subsets.

Proposition 2.4. *If X is an almost convex subset of a locally convex topological vector space E, then it is admissible.*

Proof. Let K be any nonempty compact subset of X and V be any neighborhood of the origin 0. Since E is locally convex, we may assume that V is convex and open. Noting that $K \subseteq \cup_{x\in K}\left(x+\frac{1}{2}V\right)$ and K is compact, there is a finite subset $\{x_1,\cdots,x_n\}$ of K such that

$$K \subseteq \bigcup_{i=1}^{n}\left(x_i+\frac{1}{2}V\right). \tag{1}$$

By the almost convexity of X, there is $\{y_1,\cdots,y_n\} \in \langle X\rangle$ satisfying that

$$x_i \in y_i+\frac{1}{2}V, \quad \forall i=1,\cdots,n, \text{ and} \tag{2}$$

$$co\{y_i : i=1,\cdots,n\} \subseteq X. \tag{3}$$

It follows from (1) and (2) that

$$K \subseteq \bigcup_{i=1}^{n}\left(y_i+V\right). \tag{4}$$

Let $L=\text{span}\{y_i : i=1,\cdots,n\}$ and for each $i=1,\cdots,n$, define $\rho_i : E \to \mathbb{R}$ by $\rho_i(x)=\max\{1-\mu_V(x-y_i),0\}$, where μ_V is the Minkowski functional of V, that is, $\mu_V(x)=\inf\{t>0 : t^{-1}x\in V\}$ for $x\in E$. Clearly, each ρ_i is continuous and $\rho_i(x)=0$ for $x\notin y_i+V$; $0<\rho_i(x)\leq 1$ for $x\in y_i+V$. For each $x\in E$, put $\alpha(x)=\sum_{j=1}^{n}\rho_j(x)$ and define $h : K\to E$ by $h(x)=\sum_{i=1}^{n}\frac{\rho_i(x)}{\alpha(x)}y_i$. Noting that (4) shows that for each $x\in K$, there is $j\in\{1,\cdots,n\}$ such that $\rho_j(x)>0$, we see that h is well-defined. It is obvious that h is continuous and from (3) follows that $h(K)\subseteq X\cap L$. Moreover, for all $x\in K$, since

$$\mu_V(x-h(x))=\sum_{i=1}^{n}\frac{\rho_i(x)}{\alpha(x)}\mu_V(x-y_i)<1,$$

we infer that $x-h(x)\in V$. This completes the proof. □

Corollary 2.5 (Nagumo[8]). *Any convex subset of a locally convex topological vector space is admissible.*

For other properties of admissible sets, we refer readers to Hadžić [2], Weber [11] and references therein.

3. Fixed Point Theorems

We begin with a lemma.

Lemma 3.1. *Let X, Y and Z be nonempty subsets of a topological vector space E. If $T \in SKKM(X,Y)$ and $f \in C(Y,Z)$, then $fT \in SKKM(X,Z)$.*

Proof. Let H be a generalized KKM mapping with respect to fT. For any $A = \{x_1, \cdots, x_n\} \in \langle X \rangle$, by definition there is $B = \{y_1, \cdots, y_n\} \in \langle X \rangle$ satisfying that
(i) $co(B) \subseteq X$; and
(ii) $fT\left(co\{y_i : i \in I\}\right) \subseteq \cup_{i \in I} H(x_i)$ for any nonempty subset I of $\{1, \cdots, n\}$.
Hence it follows from

$$\begin{aligned} T\left(co\{y_i : i \in I\}\right) &\subseteq f^{-1}\left(\cup_{i \in I} H(x_i)\right) \\ &= \cup_{i \in I} f^{-1}\left(H(x_i)\right) \end{aligned}$$

that $f^{-1}H$ is a generalized KKM mapping with respect to T, which in conjunction with $T \in \text{SKKM}(X,Y)$ shows that the family $\{\overline{f^{-1}(H(x))} : x \in X\}$ has the finite intersection property. Noting that $\{\overline{f^{-1}(H(x))} : x \in X\} \subseteq \{f^{-1}(\overline{H(x)}) : x \in X\}$, we see that $\{f^{-1}(\overline{H(x)}) : x \in X\}$ also has the finite intersection property, and so does the family $\{\overline{H(x)} : x \in X\}$. This shows that $fT \in \text{SKKM}(X,Z)$. □

We now in a position to prove the main result of this paper.

Theorem 3.2. *Let X be an admissible almost convex subset of a topological vector space E. If $T \in SKKM(X,X)$ is compact and closed, then it has a fixed point.*

Proof. Let $\mathcal{N}$ be a fundamental system of neighborhoods of the origin 0 in E and $V \in \mathcal{N}$. Since $\overline{T(X)}$ is a compact subset of the admissible subset X, there exists a continuous function $h : \overline{T(X)} \to X$ and a finite dimensional subspace L of E such that $x - h(x) \in V$ for all $x \in \overline{T(X)}$ and $h(\overline{T(X)}) \subseteq L \cap X$. Let $P = h(\overline{T(X)})$. By Lemma 3.1, we have that $hT \in \text{SKKM}(X,P)$. Since L is finite dimensional, it has a fundamental system $\mathcal{B}$ of neighborhoods of the origin consisting of open convex subsets, say $\mathcal{B} = \{U_\alpha : \alpha \in \Lambda\}$. Put $S = hT$. We at first claim that for each $\alpha \in \Lambda$ there is $x_\alpha \in X$ such that $(x_\alpha + U_\alpha) \cap S(x_\alpha) \neq \varnothing$. If not, then there is $U \in \mathcal{B}$ such that

$$(x + U) \cap S(x) = \varnothing, \quad \forall x \in X \tag{5}$$

Let K be the closure of $S(X)$ in E. Obviously, K is a compact subset of P. Define $G : X \to 2^P$ by $G(x) = K \setminus (x + U)$. If $G(x) = \varnothing$ for some $x \in X$, then $K \subseteq x+U$, which implies that $S(x) \subseteq x+U$, a contradiction to the hypothesis (5). So each $G(x)$ is nonempty, that is, $G : X \multimap P$. Furthermore, G is a generalized KKM mapping with respect to S. To see this, assume the contrary, that is, there is $A = \{x_1, \cdots, x_n\} \in \langle X \rangle$ such that for any $B = \{y_1, \cdots, y_n\} \in \langle X \rangle$ with $co(B) \subseteq X$, one has that $S\left(co\{y_i : i \in I\}\right) \nsubseteq \cup_{i\in I} G(x_i)$ for some nonempty subset I of $\{1, \cdots, n\}$. Since X is almost convex, there is $\{z_1, \cdots, z_n\} \in \langle X \rangle$ such that $co\{z_1, \cdots, z_n\} \subseteq X$ and

$$x_i - z_i \in \frac{1}{2}U, \quad \forall i = 1, \cdots, n. \tag{6}$$

Choose a nonempty subset I of $\{1, \cdots, n\}$ such that

$$S\left(co\{z_i : i \in I\}\right) \nsubseteq \bigcup_{i\in I} G(x_i),$$

and then choose $\mu \in co\{z_i : i \in I\}$ and $\zeta \in S(\mu)$ so that $\zeta \notin \bigcup_{i\in I} G(x_i)$. Then $\zeta \in x_i + \frac{1}{2}U$ for any $i \in I$, and so, in view of (6), $\zeta \in z_i + U$ for any $i \in I$, which implies that $\zeta \in q + U$ for any $q \in co\{z_i : i \in I\}$. In particular, $\zeta \in \mu + U$. But then $\zeta \in (\mu + U) \cap S(\mu)$, a contradiction to (5). Therefore, we conclude that G is a generalized KKM mapping with respect to S. Now since $S \in \mathrm{SKKM}(X, P)$ and each $G(x)$ is a closed subset of the compact set K, we infer that $\cap_{x\in X} G(x) \neq \varnothing$. Choose $\xi \in \cap_{x\in X} G(x)$. Since $\xi \in P \subseteq X$, we have $\xi \in G(\xi) = K \setminus (\xi + U)$, from which follows that $\xi \notin \xi + U$, a contradiction. Thus we have proved that $(x_\alpha + U_\alpha) \cap S(x_\alpha) \neq \varnothing$ for each $\alpha \in \Lambda$.

Next, for each $\alpha \in \Lambda$, choose $y_\alpha \in (x_\alpha + U_\alpha) \cap S(x_\alpha)$. Since $\{y_\alpha\}_{\alpha\in\Lambda} \subseteq P$ and P is compact, we may assume that $\{y_\alpha\}_{\alpha\in\Lambda}$ converges to some $x_V \in P$, and then $\{x_\alpha\}_{\alpha\in\Lambda} \subseteq X$ also converges to x_V. The closedness of T implies that $x_V \in S(x_V) = h(T(x_V))$. Choose $z_V \in T(x_V)$ such that $x_V = h(z_V)$. Noting that $z_V - h(z_V) \in V$, we obtain that $z_V \in h(z_V) + V = x_V + V$, so $(x_V + V) \cap T(x_V) \neq \varnothing$ for any $V \in \mathcal{N}$, which, just as before, implies that T has a fixed point. This completes the proof. □

Here, we like to give a concrete example for the above theorem. For $0 < p < 1$, let E be the topological vector space $L^P[0, 1]$ on which its topology is induced by the metric $d(f, g) = \int_0^1 |f(t) - g(t)|^p dt$ for $f, g \in L^P[0, 1]$. Put $X = E \setminus \{0\}$ and define $T : X \multimap X$ by $T(x) = C$ for each $x \in X$, where C is any nonempty closed subset of E with $0 \notin C$. Obviously, X is almost convex. We now show that it is admissible. Let K be any nonempty compact subset of X. Since $0 \notin K$

and K is compact, there is an open neighborhood W of the origin 0 such that $0 \notin K + W$. For any open neighborhood U of the origin 0, let $V = U \cap W$. By the admissibility of E (cf.[2]), there is a continuous function $h : K \to E$ such that $x - h(x) \in V \subseteq U$ for all $x \in K$, and $h(K)$ is contained in a finite-dimensional subspace L of E. It follows from $x - h(x) \in V$ for all $x \in K$ that $h(K) \subseteq K + V$, which in conjunction with $0 \notin K + W$ shows that $0 \notin h(K)$, that is, $h : K \to X$. This shows that X is admissible. It is easy to check that T is in SKKM(X, X), compact and closed. To the best of our knowledge, there is no theorem in the literature can cover this example.

If the topological vector space is locally convex, then we see that the Theorem 4.4 of [5] follows from Theorem 3.2 and Proposition 2.4.

For two topological spaces X and Y, an upper semi-continuous multimap $T : X \multimap Y$ is said to be a Kakutani multimap if either T is single-valued (in which case, Y is simply assumed to be a topological space), or $T(x)$ is a compact and convex subset of Y for any x in X (in which case, Y is assumed to be a subset of a topological vector space E). Let $\mathcal{K}$ be the class of Kakutani multimaps and $\mathcal{K}_c$ the class of finite composites of multimaps in $\mathcal{K}$, that is, $T \in \mathcal{K}_c$ if there exist two topological spaces X and Y such that
(a) $T : X \multimap Y$;
(b) $T = T_n T_{n-1} \cdots T_0$, where $T_i \in \mathcal{K}$ for $i = 0, \cdots, n$.

Lemma 3.3 ([5], Proposition 3.11). *Suppose X is a nonempty subset of a topological vector space E and $T \in \mathcal{K}_c(X, E)$. Then $T \in$ SKKM(X, E).*

Corollary 3.4. *Let X be an admissible almost convex subset of a topological vector space E. If $T : X \multimap X \in \mathcal{K}_c(X, X)$ is compact, then T has a fixed point.*

Proof. Since $T \in \mathcal{K}_c(X, X)$, it is u.s.c. with closed values, and so T is closed. Moreover, it follows from Lemma 3.3 that $T \in$ SKKM(X, X). Therefore, T has a fixed point by Theorem 3.2. $\square$

Corollary 3.5. *Let X be an admissible almost convex subset of a topological vector space E and Y a nonempty compact subset of a topological vector space F. If $T : X \multimap Y$ is closed with convex-values and $f \in \mathcal{C}(Y, X)$, then $f \circ T$ has a fixed point.*

Proof. Since T is closed and Y is compact, T is u.s.c., so in conjunction with

T being compact with convex-values, $T \in \mathcal{K}(X, Y)$. Consequently, $f \circ T$ is in $\mathcal{K}_c(X, X)$ and is compact. Hence $f \circ T$ has a fixed point by Corollary 3.4. $\square$

Lemma 3.6. *If X and Y are two admissible almost convex subsets of topological vector spaces E and F respectively, then $X \times Y$ is an admissible almost convex subset of $E \times F$.*

Proof. That $X \times Y$ is almost convex was proved in [5]. Now, suppose A is any nonempty compact subset of $X \times Y$. Both of $\text{Pr}_1(A)$ and $\text{Pr}_2(A)$ are compact in X and Y respectively, where Pr denotes the projection. We have $A \subseteq \text{Pr}_1(A) \times \text{Pr}_2(A)$. Let $V = V_1 \times V_2$ be a neighborhood of 0 in $E \times F$. Since X is admissible in E, there is a continuous function $h_1 : \text{Pr}_1(A) \to X$ such that $x - h_1(x) \in V_1$ for any $x \in \text{Pr}_1(A)$ and $h_1(\text{Pr}_1(A))$ is contained in a finite dimensional subspace L_1 of E. Similarly, there is a continuous function $h_2 : \text{Pr}_2(A) \to Y$ such that $y - h_2(y) \in V_2$ for any $y \in \text{Pr}_2(A)$ and $h_2(\text{Pr}_2(A))$ is contained in a finite dimensional subspace L_2 of F. Define $h : \text{Pr}_1(A) \times \text{Pr}_2(A) \to X \times Y$ by $h(x, y) = (h_1(x), h_2(y))$. It is clear that h is continuous, $(x, y) - h(x, y) = (x - h_1(x), y - h_2(y)) \in V_1 \times V_2$ and $h(\text{Pr}_1(A) \times \text{Pr}_2(A)) \subseteq L_1 \times L_2$. Therefore, $X \times Y$ is admissible. $\square$

Theorem 3.7. *Let $\{X_i\}_{i=1}^n$ be a finite family of admissible almost convex subsets, each in a topological vector space E_i, K_i a nonempty compact subset of X_i, and $T_i : X = \prod_{i=1}^n X_i \multimap K_i$ a closed map with convex values. Then there is an $\hat{x} \in X$ such that $\hat{x}_i \in T_i(\hat{x})$ for any $i = 1, \cdots, n$.*

Proof. Define $T : X \multimap K = \prod_{i=1}^n K_i$ by $T(x) = \prod_{i=1}^n T_i(x)$ for $x \in X$. Obviously, T is a compact closed map with convex-values. Since X is an admissible almost convex subet of $\prod_{i=1}^n E_i$ by Lemma 3.6, it follows from Corollary 3.4 that T has a fixed point $\hat{x} \in X$, that is, $\hat{x}_i \in T_i(\hat{x})$ for any $i = 1, \cdots, n$. $\square$

Let X be a subset of a topological vector space E and Y a subset of a topological vector space F. Given a neighborhood U of 0 in E and a neighborhood $V$of 0 in F, a function $s : X \to Y$ is said to be a (U, V)-selection of $A : X \multimap Y$ if for any $x \in X$, $s(x) \in (A[(x + U) \cap X] + V) \cap Y$. $A : X \multimap Y$ is said to be approachable if it has a continuous (U, V)-selection for any neighborhood U of 0 in E and any neighborhood $V$of 0 in F. $\mathcal{A}$ is defined to be the class of all u.s.c. approachable multimaps with compact values. $\mathcal{A}_c$ is the class of all finite composites of multimaps in $\mathcal{A}$.

Lemma 3.8(Ben-El-Mechaiekh and Deguire [1]). *Let X be a compact subset of a topological vector space E and Y a nonempty subset of a topological vector space F, Γ a nonempty closed subset of $X \times Y$. Then the following statements are equivalent.*
(a) $Gr(f) \cap \Gamma \neq \varnothing$ *for each* $f \in \mathcal{C}(X, Y)$.
(b) $Gr(A) \cap \Gamma \neq \varnothing$ *for each* $A \in \mathcal{A}_c(X, Y)$.

Theorem 3.9. *Let X be an admissible almost convex subset of a topological vector space E, and Y a compact subset of E. If $T \in SKKM(X, Y)$ is closed, then for every $A \in \mathcal{A}_c(Y, X)$, AT has a fixed point.*

Proof. Since $T \in \mathrm{SKKM}(X, Y)$ is closed and Y is compact, we see that T is compact and closed. For each $f \in \mathcal{C}(Y, X)$, it follows from Lemma 3.1 that $fT \in \mathrm{SKKM}(X, X)$. Clearly, fT is compact and closed. So fT has a fixed point by Theorem 3.2, that is, $Gr(f) \cap Gr(T^-) \neq \varnothing$. An application of Lemma 3.8 gives that $Gr(A) \cap Gr(T^-) \neq \varnothing$, which is equivalent to that AT has a fixed point. □

4. Applications

From Theorem 3.2 we have the following quasi-equilibrium theorem.

Theorem 4.1. *Let X be an admissible almost convex subset of a topological vector space E, $f : X \times X \to \mathbb{R}$ be u.s.c. and $H : X \multimap X$ be compact and closed. Assume that*
(a) *the function M defined on X by*

$$M(x) = \max_{y \in H(x)} f(x, y)$$

for $x \in X$ is l.s.c., and
(b) *the multimap $T : X \multimap X$ defined by*

$$T(x) = \{y \in H(x) : f(x, y) = M(x)\}$$

is in $\mathrm{SKKM}(X, X)$.
Then there exists an $\hat{x} \in X$ such that $\hat{x} \in H(\hat{x})$ and $f(\hat{x}, \hat{x}) = M(\hat{x})$.

Proof. Since f is u.s.c. and $H(x)$ is compact, $T(x)$ is nonempty. Also, let K be a compact subset of X such that $H(X) \subseteq K$. We show that $Gr(T)$ is closed in $X \times K$. To see this, let $\{(x_\alpha, y_\alpha)\}$ be a net in $Gr(T)$ and $(x_\alpha, y_\alpha) \to (x, y)$. We have

$$\begin{aligned} f(x,y) \geq \overline{\lim}_\alpha f(x_\alpha, y_\alpha) &= \overline{\lim}_\alpha M(x_\alpha) \\ &\geq \underline{\lim}_\alpha M(x_\alpha) \\ &\geq M(x), \end{aligned}$$

where the last inequality follows from (a). Since H is closed and $y_\alpha \in H(x_\alpha)$ for any $\alpha \in \Lambda$, we see that $(x, y) \in Gr(H)$, that is, $y \in H(x)$. Thus, $y \in \{z \in H(x) : f(x, z) = M(x)\}$, which shows that T is closed. Moreover, since H is compact, so is T. Therefore T has a fixed point $\hat{x}$ in X by Theorem 3.2, that is, $\hat{x} \in H(\hat{x})$ and $f(\hat{x}, \hat{x}) = M(\hat{x})$. □

For a family of sets $\{X_i\}_{i \in I}$ and a fixed $i \in I$, we write X for $\prod_{j \in I} X_j$, and X^{-i} for the set $\prod_{j \in I, j \neq i} X_j$. If $x^{-i} \in X^{-i}$ and $j \in I$ with $j \neq i$, the jth coordinate of x^{-i} is denoted by x_j^{-i}. If $x_i \in X_i$ and $x^{-i} \in X^{-i}$, then $[x_i, x^{-i}]$ is the point of X defined as follows: its ith coordinate is x_i, and for the other jth coordinate is x_j^{-i}. Obviously, any $x \in X$ can be written as $x = [x_i, x^{-i}]$ for any $i \in I$, where x^{-i} denotes the projection of x onto X^{-i}. To any multimap $G_i : X^{-i} \multimap X_i$, we associate the subset $\tilde{G}_i$ of X defined by $\tilde{G}_i = \{[x_i, x^{-i}] : x_i \in G_i(x^{-i})\}$.

Applying Theoren 3.7 and following the same argument as above, we obtain the following result.

Theorem 4.2. *Let $\{X_i\}_{i=1}^n$ be a finite family of admissible almost convex subsets, each in a topological vector space E_i, K_i a nonempty compact subset of X_i, $H_i : X \multimap K_i$ a closed map, and $f_i, g_i : X^{-i} \times X_i \to \mathbb{R}$ u.s.c. functions for any $i = 1, \cdots, n$. Suppose that for any $i = 1, \cdots, n$*
(a) $g_i(x) \leq f_i(x) \quad \forall x \in X$,
(b) *the function M_i defined on X by*

$$M_i(x) = \max_{y \in H_i(x)} g_i(x^{-i}, y)$$

is l.s.c., and
(c) *For any $x \in X$, the set $\{y \in H_i(x) : f_i(x^{-i}, y) \geq M_i(x)\}$ is convex.*
Then there exists an $\hat{x} \in X$ such that for any $i = 1, \cdots, n$, $\hat{x}_i \in H_i(\hat{x})$ and $f_i(\hat{x}^{-i}, \hat{x}_i) \geq M_i(\hat{x})$.

Proof. For each $i = 1, \cdots, n$, define $T_i : X \to 2^{K_i}$ by

$$T_i(x) = \{y \in H_i(x) : f_i(x^{-i}, y) \geq M_i(x)\}.$$

Since $H_i(x)$ is compact and $g_i(x^{-i},\ )$ is u.s.c. on $H_i(x)$, we see that $M_i(x) = g_i(x^{-i}, y_0)$ for some $y_0 \in H_i(x)$, and so by (a) $T_i(x)$ is nonempty, that is, $T_i : X \multimap K_i$. Just as in the proof of Theorem 4.1, $Gr(T_i)$ is closed in $X \times K_i$. This in conjunction with (c) shows that there is $\hat{x} \in X$ such that $\hat{x} \in T_i(\hat{x})$ for any $i = 1, \cdots, n$ by Theorem 3.7. Hence, $\hat{x_i} \in H_i(\hat{x})$ and $f_i(\hat{x}^{-i}, \hat{x_i}) \geq M_i(\hat{x})$ for any $i = 1, \cdots, n$. □

In what follows, $\mathbb{Z}_{n+1}$ denotes the set $\{0, 1, \cdots, n\}$ with addition modulo $n+1$.

Theorem 4.3. *Let X_0 be a nonempty almost convex subset of a topological vector space E_0, and X_i $(i = 1, \cdots, n)$ be a nonempty admissible almost convex subset of a topological vector space E_i. For $i = 0, \cdots, n$, suppose $G_i \in \mathcal{K}_c(X^{-i}, X_i)$ and all the multimaps G_i are compact except possibly G_n. Then $\bigcap_{i=0}^{n} \tilde{G}_i \neq \varnothing$.*

Proof. For $i \in \mathbb{Z}_{n+1}$, define $\Gamma_i : X^{-i} \multimap X^{-(i+1)}$ by

$$\Gamma_i(x^{-i}) = G_i(x^{-i}) \times \Pi_{\substack{j \in \mathbb{Z}_{n+1} \\ j \notin \{i, i+1\}}} \left\{x_j^{-i}\right\} \text{ for } x^{-i} \in X^{-i}.$$

It is easy to see that $\Gamma_i \in \mathcal{K}_c\left(X^{-i}, X^{-(i+1)}\right)$ for each $i \in \mathbb{Z}_{n+1}$. Therefore, the multimap $\Gamma : X^{-0} \multimap X^{-0}$ defined by $\Gamma = \Gamma_n \Gamma_{n-1} \cdots \Gamma_0$ belongs to $\mathcal{K}_c\left(X^{-0}, X^{-0}\right)$. Moreover, it is compact. Indeed, for $i = 0, \cdots, n-1$, since G_i is compact, there is a compact subset K_i of X_i such that $G_i(X^{-i}) \subseteq K_i \subseteq X_i$. So,

$$\Gamma_0(X^{-0}) \subseteq K_0 \times X_2 \times \cdots \times X_n,$$
$$\Gamma_1\Gamma_0(X^{-0}) \subseteq K_0 \times K_1 \times X_3 \times \cdots \times X_n,$$

and, finally, $\Gamma_{n-1}\Gamma_{n-2} \cdots \Gamma_0(X^{-0}) \subseteq K_0 \times K_1 \times \cdots \times K_{n-1}$. Hence, $\Gamma(X^{-0})$ is contained in the compact set $\Gamma_n(K_0 \times K_1 \times \cdots \times K_{n-1})$, which shows that Γ is compact. By Lemma 3.6, X^{-0} is an admissible almost convex set. So we can apply Corollary 3.4 to derive the existence of a point $x^{-0} \in X^{-0}$ such that $x^{-0} \in \Gamma(x^{-0})$. In other words, there exist $x^{-1} \in X^{-1}, \cdots, x^{-n} \in X^{-n}$ such that $x^{-(i+1)} \in \Gamma_i(x^{-i})$ for each $i \in \mathbb{Z}_{n+1}$, which means that

$$x_i^{-(i+1)} \in G_i(x^{-i}) \text{ for each } i \in \mathbb{Z}_{n+1} \tag{7}$$

and

$$x_j^{-(i+1)} = x_j^{-i} \text{ for each } j \in \mathbb{Z}_{n+1}, j \notin \{i, i+1\}. \tag{8}$$

From (8), it follows that $x_j^{-i} = x_j^{-k}$ for any $i, j, k \in \mathbb{Z}_{n+1}$ with $j \notin \{i, k\}$. Hence, $[x_i^{-(i+1)}, x^{-i}] = [x_k^{-(k+1)}, x^{-k}]$ for any $i, k \in \mathbb{Z}_{n+1}$. Denote by x the point of X defined by $x = [x_i^{-(i+1)}, x^{-i}]$ for any $i \in \mathbb{Z}_{n+1}$. From (7), we derive that $x \in \tilde{G}_i$ for every $i \in \mathbb{Z}_{n+1}$. Hence $\bigcap_{i=0}^{n} \tilde{G}_i \neq \varnothing$. $\square$

A generalized game is a game in which each player select a strategy in a subset determined by the strategies chosen by other players. Let $\mathbb{Z}_{n+1} = \{0, \cdots, n\}$ denote the set of players, and for $i \in \mathbb{Z}_{n+1}$, let X_i denote the set of strategies of the ith player. Each element of $X = \prod_{i \in \mathbb{Z}_{n+1}} X_i$ determines an outcome. The payoff to the ith player is a real-valued continuous function f_i defined on X. Given $x^{-i} \in X^{-i}$ (the strategies of all others), the choice of the ith player is restricted to a nonempty compact subset $F_i(x^{-i})$ of X_i; the ith player chooses $x_i \in F_i(x^{-i})$ so as to maximize $f_i([x_i, x^{-i}])$. An equilibrium point in such a generalized game is a strategy vector $x \in X$ such that for all $i \in \mathbb{Z}_{n+1}$, $x_i \in F_i(x^{-i})$ and $f_i(x) = \max_{y_i \in F_i(x^{-i})} f_i([y_i, x^{-i}])$.

For a nonempty subset X of a linear space E, a function $\varphi : X \to \mathbb{R}$ is said to be quasi-concave if for any $\lambda \in \mathbb{R}$, the set $\{x \in X : \varphi(x) \geq \lambda\}$ is convex.

Theorem 4.4. *Let X_0 be a nonempty almost convex subset of a topological vector space E_0, and X_i $(i = 1, \cdots, n)$ be a nonempty admissible almost convex subset of a topological vector space E_i. For $i = 0, \cdots, n$, let $F_i : X^{-i} \multimap X_i$ be a l.s.c. multimap in $\mathcal{K}(X^{-i}, X_i)$ and let $f_i : X = \prod_{i=0}^{n} X_i \to \mathbb{R}$ be a continuous function such that for any fixed $x^{-i} \in X^{-i}$, the function $x_i \to f_i([x_i, x^{-i}])$ is quasi-concave on X_i. If all the multimaps F_i are compact except possibly F_n, then there is an equilibrium point.*

Proof. For $i \in \mathbb{Z}_{n+1}$, define $G_i : X^{-i} \multimap X_i$ by

$$G_i(x^{-i}) = \left\{ x_i \in F_i(x^{-i}) : f_i([x_i, x^{-i}]) = \max_{y_i \in F_i(x^{-i})} f_i([y_i, x^{-i}]) \right\}.$$

Noting that an equilibrium point is a point of the intersection $\bigcap \left\{ \tilde{G}_i : i \in \mathbb{Z}_{n+1} \right\}$, the theorem is proved if we show that the multimaps G_i satisfy the assumptions of Theorem 4.3, that is, we have to show that each G_i is u.s.c. and each G_i is compact except possibly G_n. Let $i \in \mathbb{Z}_{n+1}$ be fixed. For any fixed $x^{-i} \in X^{-i}$, since the function $x_i \to f_i([x_i, x^{-i}])$ is continuous and quasi-concave on the nonempty compact convex set $F_i(x^{-i})$, the set $G_i(x^{-i})$ is nonempty, compact, and convex.

Now define $T_i : X^{-i} \multimap X_i$ by

$$T_i(x^{-i}) = \left\{ x_i \in X_i : f_i([x_i, x^{-i}]) \geq \max_{y_i \in F_i(x^{-i})} f_i([y_i, x^{-i}]) \right\}.$$

Obviously, $G_i = F_i \cap T_i$. Thus, if T_i is shown to be closed, then it follows from Lemma 1.2(b) that G_i is u.s.c. Since f_i is continuous on X, the functions h, $g : X \times X_i \to \mathbb{R}$ defined by $h(x, y_i) = f_i(x)$ and $g(x, y_i) = f_i(y_i, x^{-i})$ for each $(x, y_i) \in X \times X_i$ are continuous, and so the set

$$U_i = \big\{(x, y_i) \in X \times X_i : f_i(x) < f_i([y_i, x^{-i}])\big\}$$

is open in $X \times X_i$. But,

$$\begin{aligned} X \setminus \tilde{T}_i &= \big\{x \in X : \text{ there exists } y_i \in F_i(x^{-i}) \text{ such that } (x, y_i) \in U_i\big\} \\ &= \big\{x \in X : \big(\{x\} \times F_i(x^{-i})\big) \cap U_i \neq \varnothing\big\} \end{aligned}$$

is open in X because the multimap $x \to \{x\} \times F_i(x^{-i})$ from X to $X \times X_i$ is l.s.c. Thus T_i is closed and hence G_i is u.s.c. Finally, noting that each $G_i(X^{-i})$ is contained in $F_i(X^{-i})$ and each F_i is compact except possibly F_n, we see that each G_i is compact except possibly G_n. This completes the proof. □

When each X_i $(i = 1, \cdots, n)$ is almost convex in a locally convex topological vector space E_i, the above Theorems 4.3 and 4.4 are reduced to Theorems 5.1 and 5.2 of [5] respectively.

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