Fixed Point Theorems for Compact Multimaps on Almost Γ-Convex Sets in Generalized Convex Spaces

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In this paper, we introduce the concept of strict KKM property and investigate the fixed point problem for multimaps having this property on almost Γ-convex subsets of locally G-convex uniform spaces. Our new fixed point theorem generalizes the well-known Fan-Glicksberg fixed point theorem and partially extends the Himmelberg fixed point theorem. Its application to minimax theorem is also given.

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1. Introduction and Preliminaries

For two topological spaces X and Y, an upper semi-continuous multimap $T : X \multimap Y$ is said to be a Kakutani multimap if either T is single-valued (in which case, Y is simply assumed to be a topological space), or $T(x)$ is a compact and convex subset of Y for any x in X (in which case, Y is assumed to be a subset of a topological vector space). The Himmelberg fixed point theorem[5], a generalization of the famous Fan-Glicksberg fixed point theorem[3, 4], says that a compact Kakutani multimap $T : X \multimap X$ on a nonempty convex subset of a locally convex topological vector space E has a fixed point.

The main purpose of this paper is to deal with the fixed point problem on almost Γ-convex subsets in locally G-convex uniform spaces instead of convex subsets in locally convex topological vector spaces. In section 2, we start with generalized KKM mappings on a G-convex space E to introduce the concept of the strict KKM property and then show that a Kakutani multimap has the strict KKM property. The fixed point problem for multimaps having the strict KKM property is investigated in section 3, where we established a new fixed point theorem by showing that for an almost Γ-convex subset X of a locally G-convex uniform space E, any compact and closed multimap T having the strict KKM property has a fixed point. An application to minimax theorem of our fixed ponint theorem is given in section 4.

We now recall some basic definitions and facts. For a nonempty set Y, 2^Y denotes the class of all subsets of Y and $\langle Y \rangle$ denotes the class of all nonempty finite subsets of Y. A multimap $T : X \to 2^Y$ is a function from a set X into the power set 2^Y of Y. The notation $T : X \multimap Y$ stands for a multimap $T : X \to 2^Y$ having nonempty values.

In the sequel, for $n \geq 0$, Δ_n denotes the standard n-simplex of $\mathbb{R}^{n+1}$, that is,

$$\Delta_n = \left\{ \alpha = (\alpha_0, \cdots, \alpha_n) \in \mathbb{R}^{n+1} : \alpha_i \geq 0 \text{ for all } i \text{ and} \sum_{i=0}^{n} \alpha_i = 1 \right\};$$

and $\{\mathbf{e}_0, \cdots, \mathbf{e}_n\}$, the standard basis of $\mathbb{R}^{n+1}$, is the set of the vertices of Δ_n.

If X and Y are two subsets of a linear space E, a multimap $F : X \multimap Y$ satisfying $\mathrm{co}(A) \subseteq F(A)$ for any $A \in \langle X \rangle$ is called a KKM mapping, where $\mathrm{co}(A)$

denotes the convex hull of A. The most important result for KKM mapping is the KKM Lemma published in 1929 due to Knaster, Kuratowski and Mazurkiewicz:

KKM Lemma 1.1.(cf.[1, 12]) *Suppose $F_0, \cdots, F_n$ are closed subsets of the standard n-simplex Δ_n in $\mathbb{R}^{n+1}$. If for any nonempty subset I of $\{0, \cdots, n\}$, $co\{\mathbf{e}_i : i \in I\} \subseteq \bigcup_{i\in I} F_i$, then $\bigcap_{i=0}^{n} F_i \neq \varnothing$.*

For a multimap $T : X \to 2^Y$, $A \subseteq X$ and $B \subseteq Y$, the image of A under T is the set $T(A) = \bigcup_{x\in A} T(x)$; and the inverse image of B under T is $T^-(B) = \{x \in X : T(x) \cap B \neq \varnothing\}$.

All topological spaces are supposed to be Hausdorff. The closure of a subset X of a topological space is denoted by $\overline{X}$. Let X and Y be two topological spaces. A multimap $T : X \to 2^Y$ is said to be

(a) upper semicontinuous(u.s.c.) if $T^-(B)$ is closed in X for each closed subset B of Y;

(b) lower semicontinuous(l.s.c.) if $T^-(B)$ is open in X for each open subset B of Y;

(c) compact if $T(X)$ is contained in a compact subset of Y;

(d) closed if its graph $\mathrm{Gr}(T) = \{(x, y) : y \in T(x), x \in X\}$ is a closed subset of $X \times Y$.

Lemma 1.2.(Lassonde[9, Proposition 1]) *Let X, Y be topological spaces and $T : X \multimap Y$.*

(a) *If Y is regular and T is u.s.c. with closed values, then T is closed.*

(b) *If $T : X \multimap Y$ is u.s.c. with compact values and $S : X \multimap Y$ is closed, then $T \cap S : X \multimap Y$, defined by $(T \cap S)(x) = T(x) \cap S(x)$ for each $x \in X$, is u.s.c.; in particular, if S is compact and closed, then S is u.s.c.*

2. The Strict KKM Property

Definition 2.1. *A generalized convex space or a G-convex space $(E; \Gamma)$ consists of a topological space E and a map $\Gamma : \langle E\rangle \multimap E$ such that*

(a) *for any* $A, B \in \langle E\rangle$, $A \subseteq B$ *implies* $\Gamma(A) \subseteq \Gamma(B)$*; and*

(b) *for each* $A = \{a_0, \cdots, a_n\} \in \langle E\rangle$ *with* $|A| = n+1$*, there exists a continuous function* $\varphi_A : \Delta_n \to \Gamma(A)$ *such that if* $0 \leq i_0 < i_1 < \cdots < i_k \leq n$*, then* $\varphi_A(co\{\mathbf{e}_{i_0}, \cdots, \mathbf{e}_{i_k}\}) \subseteq \Gamma(\{a_{i_0}, \cdots, a_{i_k}\})$.

In this paper, we assume that a G-convex space $(E;\Gamma)$ *always satisfies the extra condition:* $x \in \Gamma(\{x\})$ *for any* $x \in E$.

A subset K *of a G-convex space* $(E;\Gamma)$ *is said to be* Γ*-convex if for any* $A \in \langle K\rangle$, $\Gamma(A) \subseteq K$. *For a nonempty subset* Q *of* E*, the* Γ*-convex hull of* Q*, denoted by* Γ*-co(Q), is defined by*

$$\Gamma\text{-}co(Q) = \cap\{C : Q \subset C \subset E, C \text{ is } \Gamma\text{-convex}\}.$$

It is easy to see that Γ-co(Q) is the smallest Γ-convex subset of E containing Q.

For convenience, we also express $\Gamma(A)$ by Γ_A.

A uniformity for a set X is a nonempty family $\mathcal{U}$ of subsets of $X \times X$ such that

(a) each member of $\mathcal{U}$ contains the diagonal Δ;

(b) if $U \in \mathcal{U}$, then $U^{-1} \in \mathcal{U}$;

(c) if $U \in \mathcal{U}$, then $V \circ V \subseteq U$ for some V in $\mathcal{U}$;

(d) if U and V are members of $\mathcal{U}$, then $U \cap V \in \mathcal{U}$; and

(e) if $U \in \mathcal{U}$ and $U \subseteq V \subseteq X \times X$, then $V \in \mathcal{U}$.

Every member V in $\mathcal{U}$ is called an entourage. An entourage V is said to symmetric if $(x, y) \in V$ whenever $(y, x) \in V$.

If $(X,\mathcal{U})$ is a uniform space, then the topology $\mathcal{T}$ induced by $\mathcal{U}$ is the family of all subsets W of X such that for each x in W there is U in $\mathcal{U}$ such that $U[x] \subseteq W$, where $U[x]$ is defined as $\{y \in X : (x, y) \in U\}$. If H is a subset of X and U is in $\mathcal{U}$, then $U(H) := \cup_{x\in H} U[x]$. For details of uniform spaces we refer to [8].

Definition 2.2. *A G-convex uniform space* $(E;\mathcal{U},\Gamma)$ *is a G-convex space so that* *its topology is induced by a uniformity* $\mathcal{U}$. *A G-convex uniform space* $(E;\mathcal{U},\Gamma)$ *is said to be a locally G-convex uniform space if the uniformity* $\mathcal{U}$ *has a base* $\mathcal{B}$ *consisting of open symmetric entourages such that for each* $V \in \mathcal{B}$ *and any* $x \in E$, $V[x] := \{y \in X : (x,y) \in V\}$ *is* Γ*-convex.*

By the definition of a G-convex uniform space $(E;\mathcal{U},\Gamma)$, it is easy to check that $A \subseteq \Gamma_A$ for any $A \in \langle E \rangle$.

Motivated by the works of Chang and Zhang[2], we make the following definition.

Definition 2.3. *Let* X *be a nonempty subset of a G-convex space* $(E;\Gamma)$ *and* $F : X \multimap E$. *If* F *satisfies that for any* $\{x_1,\cdots,x_n\} \in \langle X \rangle$, *there is* $\{y_1,\cdots,y_n\} \in \langle X \rangle$ *such that*

$$\Gamma_{\{y_i : i \in I\}} \subseteq \bigcup_{i \in I} F(x_i)$$

for any nonempty subset I *of* $\{1,\cdots,n\}$, *then* F *is called a generalized KKM mapping. If* $\Gamma_A \subseteq F(A)$ *for each* $A \in \langle E \rangle$, *then* F *is called a KKM mapping.*

It is easy to see that a KKM mapping is a generalized KKM mapping by putting $y_i = x_i (i = 1,\cdots,n)$ for any $\{x_1,\cdots,x_n\} \in \langle X \rangle$. However, a generalized KKM mapping may not be a KKM mapping as shown in [7].

Proposition 2.4. *Let* X *be a nonempty subset of a G-convex space* $(E;\Gamma)$. *If* $F : X \multimap E$ *is a generalized KKM mapping, then* $\{\overline{F(x)} : x \in X\}$ *has the finite intersection property.*

Proof. For any $\{x_0,x_1,\cdots,x_n\} \in \langle X \rangle$, since F is a generalized KKM mapping, there is $\{y_0,y_1,\cdots,y_n\} \in \langle X \rangle$ such that

$$\Gamma_{\{y_i : i \in I\}} \subseteq \bigcup_{i \in I} F(x_i)$$

for any nonempty subset I of $\{0,1,\cdots,n\}$. Let $B = \{y_i : i = 0,1,\cdots,n\}$. Since

$(E;\Gamma)$ is a G-convex space, there is a continuous function $\phi_B : \Delta_n \to \Gamma_B$ such that

$$\phi_B\left(\text{co}\{\mathbf{e}_i : i \in I\}\right) \subseteq \Gamma_{\{y_i : i\in I\}} \cap \phi_B(\Delta_n).$$

for any $I \in \langle \{0, 1, \cdots, n\}\rangle$. So,

$$\begin{aligned}\text{co}\{\mathbf{e}_i : i \in I\} &\subseteq \phi_B^-\left(\Gamma_{\{y_i:i\in I\}} \cap \phi_B(\Delta_n)\right)\\ &\subseteq \phi_B^-\left((\cup_{i\in I} F(x_i)) \cap \phi_B(\Delta_n)\right)\\ &\subseteq \phi_B^-\left(\left(\cup_{i\in I}\overline{F(x_i)}\right) \cap \phi_B(\Delta_n)\right)\\ &= \cup_{i\in I}\phi_B^-\left(\overline{F(x_i)} \cap \phi_B(\Delta_n)\right).\end{aligned}$$

By KKM lemma, $\cap_{i=0}^n \phi_B^-\left(\overline{F(x_i)} \cap \phi_B(\Delta_n)\right) \neq \varnothing$, so $\cap_{i=0}^n \phi_B^-(\overline{F(x_i)}) \neq \varnothing$. Any $z \in \cap_{i=0}^n \phi_B^-(\overline{F(x_i)})$ satisfies that $\phi_B(z) \in \cap_{i=0}^n \overline{F(x_i)}$. □

Theorem 2.5. *Let $(E;\mathcal{U},\Gamma)$ be a locally G-convex uniform space such that $\Gamma_{\{x\}} = \{x\}$ for any $x \in E$. If X is a nonempty subset of E and $F : X \multimap E$ is a multimap with closed values such that $F(x_0)$ is compact for some $x_0 \in X$, then F is a generalized KKM mapping if and only if $\bigcap_{x\in X} F(x) \neq \varnothing$.*

Proof. If F is a generalized KKM mapping, then, in view of $F(x_0)$ being compact and Proposition 2.4, $\bigcap_{x\in X} F(x) \neq \varnothing$. Conversely, if $\bigcap_{x\in X} F(x) \neq \varnothing$, then $\{F(x) : x \in X\}$ has the finite intersection property. So for any $\{x_1, \cdots, x_n\} \in \langle X\rangle$, there is $y \in \bigcap_{i=1}^n F(x_i)$. Putting $y_i = y$ for any $i = 1, \cdots, n$, we conclude that $\Gamma_{\{y_i:i\in I\}} = \Gamma_{\{y\}} = \{y\} \subseteq \bigcup_{i\in I} F(x_i)$, for any nonempty subset I of $\{1, \cdots, n\}$. Hence F is a generalized KKM mapping. □

The concept of strict KKM property on a topological vector space in [7] can be extended to that on G-convex spaces. .

Definition 2.6. *Suppose X and Y are two nonempty subsets of a G-convex space $(E;\Gamma)$, and T, $F : X \multimap Y$. We say that F is a generalized KKM mapping with respect to T if for any $A = \{x_1, \cdots, x_n\} \in \langle X\rangle$ there is $B = \{y_1, \cdots, y_n\} \in \langle X\rangle$*

satisfying
(a) $\Gamma_B \subseteq X$, *and*
(b) $T\left(\Gamma_{\{y_i:i\in I\}}\right) \subseteq \bigcup_{i\in I} F(x_i)$ *for any nonempty subset* I *of* $\{1, \cdots, n\}$.
If a multimap $T : X \multimap Y$ *satisfies that for any generalized KKM mapping* $F : X \multimap Y$ *with respect to* T, *the family* $\{\overline{F(x)} : x \in X\}$ *has the finite intersection property, then* T *is said to have the strict KKM property.*

Checking the proof of Watson [11, Lemma 1], the following lemma holds.

Lemma 2.7.(Watson[11]) *Suppose* X *is a nonempty compact subset of a locally* G*-convex uniform space* $(E;\mathcal{U},\Gamma)$, $p : X \to \Delta_n$ *is continuous and* $T : \Delta_n \multimap X$ *is u.s.c. with closed* Γ*-convex values. Then* $p \circ T : \Delta_n \multimap \Delta_n$ *has a fixed point.*

Proposition 2.8. *Suppose* X *is a nonempty subset of a locally* G*-convex uniform space* $(E;\mathcal{U},\Gamma)$ *and* $T : X \multimap E$ *is compact and u.s.c. with closed* Γ*-convex values. Then* T *has the strict KKM property.*

Proof. Assume that T does not have the strict KKM property. Then there is a closed-valued generalized KKM mapping $F : X \multimap E$ with respect to T such that $\bigcap_{i=0}^n F(x_i) = \varnothing$ for some $\{x_0, \cdots, x_n\} \in \langle X\rangle$. Choose $B = \{y_0, \cdots, y_n\} \in \langle X\rangle$ such that $\Gamma_B \subseteq X$ and $T(\Gamma_{\{y_i:i\in I\}}) \subseteq \bigcup_{i\in I} F(x_i)$ for any nonempty subset I of $\{0, 1, \cdots, n\}$. Since T is compact, the set $K := \overline{T(X)}$ is a compact subset of E. By the definition of a G-convex space, there is a continuous function

$$\varphi_B : \Delta_n \to \Gamma_B \subseteq X$$

such that $\varphi_B(\Delta_J) \subseteq \Gamma_{\{y_i:i\in J\}}$ for any $J \in \langle\{0, 1, \cdots, n\}\rangle$. Noting that $K \subseteq E = \bigcup_{i=0}^n F(x_i)^c$, there is a partition of unity $\{\alpha_i\}_{i=0}^n$ subordinated to $\{F(x_i)^c\}_{i=0}^n$. Define $p : K \to \Delta_n$ by $p(x) = \sum_{i=0}^n \alpha_i(x)\mathbf{e}_i$. It is clear that $p \circ T \circ \varphi_B : \Delta_n \multimap \Delta_n$ is u.s.c. By Watson Lemma, $p\circ T\circ\varphi_B$ has a fixed point $\hat{x}$, that is $\hat{x} \in (p\circ T\circ\varphi_B)(\hat{x})$. Choose $\hat{y} \in (T\circ\varphi_B)(\hat{x})$ so that $\hat{x} = p(\hat{y})$. Put $J = \{i \in \{0, 1, \cdots, n\} : \alpha_i(\hat{y}) > 0\}$. It is easy to see that $i \in J$ if and only if $\hat{y} \notin F(x_i)$. So $\hat{y} \notin \bigcup_{i\in J} F(x_i)$, which in view of

$$\hat{y} \in T(\varphi_B(\hat{x})) \subseteq T(\varphi_B(\Delta_J)) \subseteq T\left(\Gamma_{\{y_i : i \in J\}}\right)$$

implies that $T\left(\Gamma_{\{y_i:i\in J\}}\right) \nsubseteq \bigcup_{i\in J} F(x_i)$, a contradiction to the fact that F is a generalized KKM mapping with respect to T. Hence, T has the strict KKM property, completing the proof. □

3. Fixed Point Theorems

In this section we at first extend the concept of almost convex sets in topological vector spaces to locally G-convex uniform spaces and then investigate the fixed point problem on such sets.

Definition 3.1. *A nonempty subset X of a G-convex uniform space $(E;\mathcal{U},\Gamma)$ is said to be almost Γ-convex if for any $\{x_1,\cdots,x_n\} \in \langle X\rangle$ and for any entourage $U \in \mathcal{U}$ there is $\{y_1,\cdots,y_n\} \in \langle X\rangle$ such that $y_i \in U[x_i]$ for each $i \in \{1,\cdots,n\}$ and Γ-co$(\{y_1,\cdots,y_n\}) \subseteq X$.*

Here, we like to give a concrete example of an almost Γ-convex subset of a G-convex uniform space. Let E be the closed triangle with vertices $(0,0)$, $(1,0)$ and $(0,1)$ in the Euclidean space $\mathbb{R}^2$ and let $\mathcal{U}$ be the usual Euclidean uniformity restricted to E. Define $\Gamma : \langle E\rangle \multimap E$ by $\Gamma(\{\mathbf{x}\}) = \{\mathbf{x}\}$ for each $\mathbf{x} \in E$ and $\Gamma(\{\mathbf{x}_1,\cdots,\mathbf{x}_n\}) = \cup_{i=1}^{n}[\mathbf{0},\mathbf{x}_i]$ for any finite subset $\{\mathbf{x}_1,\cdots,\mathbf{x}_n\}$ of E, where $[\mathbf{0},\mathbf{x}]$ the closed line segment joining $\mathbf{0}$ with $\mathbf{x}$. Put A to be the subset of E with the open line segment $((0,0),(1,0))$ deleted. By the concept of c-structure of Horvath [6], we see that $(E;\mathcal{U},\Gamma)$ is a G-convex uniform space. Moreover, it is easy to check that A is almost Γ-convex but not Γ-convex in E.

Proposition 3.2. *Suppose the locally G-convex uniform space $(E;\mathcal{U},\Gamma)$ satisfies the property that $V[K]$ is Γ-convex whenever K is a Γ-convex subset of E and V is any member in some base $\mathcal{B}$ consisting of open symmetric entourages of*

$\mathcal{U}$. *Then the closure $\overline{X}$ of an almost Γ-convex subset X of E is Γ-convex.*

Proof. Let $A = \{x_1, \cdots, x_n\} \in \langle \overline{X} \rangle$. We have to show that $\Gamma_A \subseteq \overline{X}$. For any $W \in \mathcal{B}$, choose a symmetric entourage V so that $V \circ V \subseteq W$; and for any $i \in \{1, \cdots, n\}$, choose a $y_i \in X \cap V[x_i]$. Since X is almost Γ-convex, there is $\{z_1, \cdots z_n\} \in \langle X \rangle$ such that for any $i = 1, \cdots, n$,

$$z_i \in V[y_i] \text{ and } \Gamma\text{-co}(\{z_1, \cdots, z_n\}) \subseteq X.$$

It follows from $(x_i, y_i) \in V$ and $(y_i, z_i) \in V$ that $(x_i, z_i) \in V \circ V$, so $z_i \in (V \circ V)[x_i]$. By symmetry, we have

$$\begin{aligned} x_i \in (V \circ V)[z_i] &\subseteq (V \circ V)[\{z_1, \cdots, z_n\}] \\ &\subseteq (V \circ V)[\Gamma\text{-co}\{z_1, \cdots, z_n\}] \\ &\subseteq W[\Gamma\text{-co}\{z_1, \cdots, z_n\}]. \end{aligned}$$

Since $W[\Gamma\text{-co}\{z_1, \cdots, z_n\}]$ is Γ-convex by assumption, we infer that

$$\Gamma_{\{x_1, \cdots, x_n\}} \subseteq W[\Gamma\text{-co}\{z_1, \cdots, z_n\}] \subseteq W[X].$$

Therefore, $\Gamma_{\{x_1, \cdots, x_n\}} \subseteq \cap_{W \in \mathcal{U}} W[X] = \overline{X}$. □

We now take up the fixed point problem on an almost Γ-convex subset of a locally G-convex uniform space. To begin with, a key lemma will be established.

Lemma 3.3. *Let X be an almost Γ-convex subset of a locally G-convex uniform space $(E; \mathcal{U}, \Gamma)$. If $T : X \multimap X$ is compact and has the strict KKM property, then for any $U \in \mathcal{U}$, there is $x_U \in X$ such that $U[x_U] \cap T(x_U) \neq \varnothing$.*

Proof. On the contrary, assume there is a $U \in \mathcal{U}$ such that

$$U[x] \cap T(x) = \varnothing \tag{1}$$

for any $x \in X$. Let $K = \overline{T(X)}$ and choose a symmetric entourage V so that $V \subseteq V \circ V \subseteq U$. By assumption, K is compact. Define $F : X \to 2^X$ by

$F(x) = K \setminus V[x]$ for each $x \in X$. Since $U[x] \cap T(x) = \varnothing$, we have that

$$\begin{aligned}\varnothing \neq T(x) &\subseteq K \setminus U[x] \\ &\subseteq K \setminus V[x] = F(x),\end{aligned}$$

so $F(x) \neq \varnothing$ for each $x \in X$, that is $F : X \multimap X$. We now show that F is a generalized KKM mapping with respect to T. If not, there exists $A = \{x_1, \cdots, x_n\} \in \langle X \rangle$ such that for any $B = \{y_1, \cdots, y_n\} \in \langle X \rangle$ with $\Gamma_B \subseteq X$, one has $T(\Gamma_{\{y_i : i \in I\}}) \nsubseteq \bigcup_{i \in I} F(x_i)$ for some nonempty subset I of $\{1, \cdots, n\}$. Since X is almost Γ-convex, there is $\{z_1, \cdots, z_n\} \in \langle X \rangle$ such that $\Gamma\text{-co}\{z_1, \cdots, z_n\} \subseteq X$ and

$$x_i \in V[z_i] \tag{2}$$

for any $i = 1, \cdots, n$. Choose a nonempty subset I of $\{1, \cdots, n\}$ such that

$$T\left(\Gamma_{\{z_i : i \in I\}}\right) \nsubseteq \bigcup_{i \in I} F(x_i),$$

and then choose $\mu \in \Gamma_{\{z_i : i \in I\}}$ and $\zeta \in T(\mu)$ so that $\zeta \notin \bigcup_{i \in I} F(x_i)$. Then $\zeta \in V[x_i]$ for any $i \in I$, and so, in view of (2), $\zeta \in (V \circ V)[z_i]$ for any $i \in I$, that is $z_i \in (V \circ V)[\zeta] \subseteq U[\zeta]$ for any $i \in I$. So, by noting that $U[\zeta]$ is Γ-convex, we infer that $\Gamma_{\{z_i : i \in I\}} \subseteq U[\zeta]$, and hence $\zeta \in U[p]$ for any $p \in \Gamma_{\{z_i : i \in I\}}$. In particular, $\zeta \in U[\mu]$. But then $\zeta \in U[\mu] \cap T(\mu)$, a contradiction to (1). Therefore, we conclude that F is a generalized KKM mapping with respect to T.

Finally, since T has the strict KKM property and F is compact-valued, the family $\{F(x) : x \in X\}$ has the finite intersection property, so $\bigcap_{x \in X} F(x) \neq \varnothing$. Choosing $\eta \in \bigcap_{x \in X} F(x)$ and noting that $\bigcap_{x \in X} F(x) = K \setminus \bigcup_{x \in X} V[x]$, we see that $\eta \notin V[\eta]$, a contradiction. Thus there is $x_U \in X$ such that $U[x_U] \cap T(x_U) \neq \varnothing$. This completes the proof. □

Theorem 3.4. *Let X be an almost Γ-convex subset of a locally G-convex uniform space $(E; \mathcal{U}, \Gamma)$. If $T : X \multimap X$ is compact, closed and has the strict KKM property, then T has a fixed point.*

Proof. Let $\mathcal{B}$ be a base of $\mathcal{U}$ consisting of open symmetric entourages. By Lemma 3.3, for any $V \in \mathcal{B}$ there is $x_V \in X$ such that $V[x_V] \cap T(x_V) \neq \varnothing$. Choose $y_V \in V[x_V] \cap T(x_V)$. Since T is compact, we may assume that $\{y_V\}_{V\in\mathcal{B}}$ converges to y_0. For any $W \in \mathcal{B}$, choose $U \in \mathcal{B}$ such that $U \circ U \subseteq W$. Since $\{y_V\}_{V\in\mathcal{B}}$ converges to y_0, there is $V_0 \in \mathcal{B}$ such that $V_0 \subseteq U$ and

$$y_V \in U[y_0], \ \forall V \in \mathcal{B} \text{ with } V \subseteq V_0,$$

that is,

$$(y_V, y_0) \in U, \ \forall V \in \mathcal{B} \text{ with } V \subseteq V_0.$$

Thus, for $V \in \mathcal{B}$ with $V \subseteq V_0$, it follows from

$$(x_V, y_V) \in V \subseteq U \text{ and } (y_V, y_0) \in U$$

that $(x_V, y_0) \in U \circ U \subseteq W$. Hence $x_V \in W[y_0]$. This shows that $\{x_V\}_{V\in\mathcal{B}}$ converges to y_0. Since T is closed, we conclude that $y_0 \in T(y_0)$, completing the proof. □

The above theorem partially extends Himmelberg [5, Theorem 2] as the following corollary shows.

Corollary 3.5. *Suppose X is an almost Γ-convex subset of a locally G-convex uniform space $(E;\mathcal{U},\Gamma)$ and $T : X \multimap E$ is compact and u.s.c. with closed Γ-convex values. If $T(X) \subseteq X$, then it has a fixed point.*

Proof. This follows immediately from Proposition 2.8 and Theorem 3.4. □

As an application of Theorem 3.4, we establish a quasi-equlibrium theorem.

Theorem 3.6. *Let X be an almost Γ-convex subset of a locally G-convex uniform space $(E;\mathcal{U},\Gamma)$, $f : X \times X \to \mathbb{R}$ be upper semi-continuous and $H : X \multimap X$ be compact and closed. Assume that*
(a) *the function M defined on X by*

$$M(x) = \max_{y\in H(x)} f(x, y)$$

for $x \in X$ is lower semi-continuous, and

(b) *the multimap $T : X \multimap X$ defined by*

$$T(x) = \{y \in H(x) : f(x, y) = M(x)\}$$

has the strict KKM property.

Then there exists an $\hat{x} \in X$ such that $\hat{x} \in H(\hat{x})$ and $f(\hat{x}, \hat{x}) = M(\hat{x})$.

Proof. Since f is upper semi-continuous and $H(x)$ is compact, $T(x)$ is nonempty. Also, let K be a compact subset of X such that $H(X) \subseteq K$. We show that $Gr(T)$ is closed in $X \times K$. To see this, let $\{(x_\alpha, y_\alpha)\}_\alpha$ be a net in $Gr(T)$ and $(x_\alpha, y_\alpha) \to (x, y)$. We have

$$\begin{aligned} f(x, y) \geq \overline{\lim}_\alpha f(x_\alpha, y_\alpha) &= \overline{\lim}_\alpha M(x_\alpha) \\ &\geq \underline{\lim}_\alpha M(x_\alpha) \\ &\geq M(x), \end{aligned}$$

where the last inequality follows from (a). Since H is closed and $y_\alpha \in H(x_\alpha)$ for any α, we see that $(x, y) \in Gr(H)$, that is, $y \in H(x)$. Thus, $y \in \{z \in H(x) : f(x, z) = M(x)\}$, which shows that T is closed. Moreover, since H is compact, so is T. Therefore T has a fixed point $\hat{x}$ in X by Theorem 3.4, that is, $\hat{x} \in H(\hat{x})$ and $f(\hat{x}, \hat{x}) = M(\hat{x})$. □

By means of Corollary 3.5, we can extend the Theorem 1 of Himmelberg [5] in the following manner.

Theorem 3.7. *Suppose the locally G-convex uniform space $(E; \mathcal{U}, \Gamma)$ satisfies the conditions:*

(a) Γ_C *is closed and* Γ*-convex for any* $C \in \langle E \rangle$,

(b) *there is a base $\mathcal{B}$ of $\mathcal{U}$ consisting of closed symmetric entourages so that for any $V \in \mathcal{B}$, $V[K]$ is Γ-convex whenever K is a Γ-convex subset of E.*

If X is a nonempty compact subset of E and $T : X \multimap X$ is an u.s.c. multimap such that $T(x)$ is closed for all $x \in X$ and Γ-convex for all x in some dense *almost* Γ*-convex subset* A *of* X*, then* T *has a fixed point.*

Proof. For each $V \in \mathcal{B}$, let $F_V = \{x \in X : x \in V[T(x)]\}$. In view of

$$\begin{aligned} x \in \bigcap_{V\in\mathcal{B}} F_V &\Leftrightarrow x \in \bigcap_{V\in\mathcal{B}} V[T(x)] \\ &\Leftrightarrow x \in \overline{T(x)} = T(x), \text{ since } \bigcap_{V\in\mathcal{B}} V[T(x)] = \overline{T(x)}, \end{aligned}$$

to find a fixed point of T it suffices to show that

$$\bigcap_{V\in\mathcal{B}} F_V \neq \varnothing. \tag{1}$$

As for (1), noting that $F_U \cap F_V \supseteq F_{U\cap V}$ and X is compact, we need only to show each F_V is closed and nonempty.

For each $V \in \mathcal{B}$, define $T_V : X \multimap X$ by

$$T_V(x) = V[T(x)] \cap X$$

for each $x \in X$. We claim that T_V is closed, that is, $Gr(T_V)$ is closed in $X \times X$. Let $(x, y) \in \overline{Gr(T_V)}$ and choose a net $\{(x_\alpha, y_\alpha)\}_\alpha$ in $Gr(T_V)$ so that $\lim_\alpha (x_\alpha, y_\alpha) = (x, y)$. By the definition of T_V, for each α, there is $z_\alpha \in T(x_\alpha)$ such that $y_\alpha \in V[z_\alpha] \cap X$. Since X is compact, there is a subnet $\{z_{\alpha_j}\}$ of $\{z_\alpha\}$ such that $z_{\alpha_j} \to z$ for some $z \in X$, and so, $(y_{\alpha_j}, z_{\alpha_j}) \to (y, z)$. In light of the closedness of V in $E \times E$, it follows that $(y, z) \in V$, that is, $y \in V[z]$. Moreover, since $x_{\alpha_j} \to x$, $z_{\alpha_j} \to z$ and T is closed, we have $z \in T(x)$, and hence, $y \in V[T(x)]$, which shows that $(x, y) \in Gr(T_V)$. Now let Δ be the diagonal in $X \times X$. Since

$$\begin{aligned} x \in F_V &\Leftrightarrow x \in V[T(x)] \cap X \\ &\Leftrightarrow x \in T_V(x) \\ &\Leftrightarrow (x, x) \in Gr(T_V) \cap \Delta \end{aligned}$$

and since $Gr(T_V) \cap \Delta$ is closed in $X \times X$, we infer that F_V is closed.

It remains to show that F_V is nonempty. Choose $W \in \mathcal{B}$ so that

$$W \circ W \circ W \circ W \subseteq V. \tag{2}$$

Since $X \subseteq \cup_{x\in X} W[x]$ and X is compact, there is a finite subset $\{x_1, \cdots, x_m\}$ of X such that

$$X \subseteq \bigcup_{i=1}^{m} W[x_i]. \tag{3}$$

By the denseness of A in X, there is a finite subset $\{y_1, \cdots, y_m\}$ of A such that

$$y_i \in W[x_i], \qquad \forall i = 1, \cdots, m. \tag{4}$$

Since A is almost Γ-convex, there is $\{z_1, \cdots, z_m\} \in \langle A\rangle$ such that

$$\Gamma\text{-co}\{z_1, \cdots, z_m\} \subseteq A \text{ and } z_i \in W[y_i], \qquad \forall i = 1, \cdots, m. \tag{5}$$

Then (4) and (5) imply that

$$(x_i, z_i) \in W \circ W, \quad \forall i = 1, \cdots, m. \tag{6}$$

Let $C = \Gamma\text{-co}\{z_1, \cdots, z_m\}$. Since $\{z_1, \cdots, z_n\} \subseteq \Gamma_{\{z_1,\cdots,z_n\}}$ and $\Gamma\text{-co}\{z_1, \cdots, z_m\}$ is the smallest Γ-convex subset containing $\{z_1, \cdots, z_n\}$, we have

$$\Gamma_{\{z_1,\cdots,z_n\}} = \Gamma\text{-co}\{z_1, \cdots, z_m\} = C,$$

and so it follows from hypothesis (a) that C is closed. Also, we have $C \subseteq A$ by (5). Define $H_V : C \to 2^C$ by $H_V(x) = T_V(x) \cap C$. In view of Lemma 1.2, H_V is u.s.c. Since both of $T_V(x)$ and C are Γ-convex and closed, so is $H_V(x)$. Moreover, $H_V(x) \neq \varnothing$. To see this, for any $y \in W[T(x)] \cap X$, choose $k \in T(x)$ such that $y \in W[k] \cap X$. By (3), there is x_j such that $y \in W[x_j]$, and so $(x_j, k) \in W \circ W$. Meanwhile, $(x_j, z_j) \in W \circ W$ by (6). Hence, $(z_j, k) \in W \circ W \circ W \circ W \subseteq V$, and so

$$\begin{aligned} z_j \in V[k] \cap C &\subseteq V[T(x)] \cap C \\ &= T_V(x) \cap C = H_V(x), \end{aligned}$$

which shows that $H_V(x) \neq \varnothing$ for each $x \in C$. Consequently, $H_V : C \multimap C$ has a fixed point $\hat{x}$ by Corollary 3.5. Then

$$\begin{aligned} \hat{x} \in H_V(\hat{x}) &= T_V(\hat{x}) \cap C \\ &= V[T(\hat{x})] \cap C, \end{aligned}$$

that is, $\hat{x} \in F_V$, so F_V is nonempty. $\square$

4. Application to Minimax Theorem

Tan and Zhang[10] showed that the product of an arbitrary family of G-convex spaces is a G-convex space: *Suppose $\{(E_i;\Gamma_i)\}_{i\in I}$ is any family of G-convex spaces. Let $E = \Pi_{i\in I}E_i$ be equipped with product topology. For each $i \in I$, let $\pi_i : E \to E_i$ be the projection. Define $\Gamma = \Pi_{i\in I}\Gamma_i : \langle E\rangle \multimap E$ by*

$$\Gamma(A) = \Pi_{i\in I}\Gamma_i(\pi_i(A))$$

for each $A \in \langle E\rangle$. Then $(E;\Gamma)$ is a G-convex space.

Lemma 4.1. *Let $(E;\Gamma) = (\Pi_{i\in I}E_i;\Pi_{i\in I}\Gamma_i)$ be the product G-convex space of a family of G-convex spaces $(E_i;\Gamma_i)$, $i \in I$. Then $K := \Pi_{i\in I}K_i$ is Γ-convex in E provided for each $i \in I$, K_i is a Γ_i-convex subset of E_i.*

Proof. Let π_i be the projection from E to E_i¿ For each $A \in \langle K\rangle$, it is clear that $\pi_i(A) \in \langle K_i\rangle$. Since K_i is Γ_i-convex, we have $\Gamma_i(\pi_i(A)) \subseteq K_i$, and so

$$\Gamma_A = \Pi_{i\in I}\Gamma_i(\pi_i(A)) \subseteq \Pi_{i\in I}K_i,$$

which shows that $\Pi_{i\in I}K_i$ is Γ-convex. $\square$

Since the topology of the product uniformity is the product topology, the following proposition holds.

Proposition 4.2. *Suppose $\{(E_i;\mathcal{U}_i,\Gamma_i)\}_{i\in I}$ is any family of locally G-convex uniform spaces. Let $E = \Pi_{i\in I}E_i$ be equipped with product topology and $\mathcal{U} = \Pi_{i\in I}\mathcal{U}_i$ be the product uniformity on E. Then $(E;\mathcal{U},\Gamma)$ is a locally G-convex uniform space.*

Proof. It suffices to show that the product uniformity $\mathcal{U}$ has a base $\mathcal{B}$ consisting of open symmetric entourages such that for each $V \in \mathcal{B}$ and for each $x \in E$, $V[x]$ is Γ-convex. For any $i \in I$, let $\mathcal{B}_i$ be a base of $\mathcal{U}_i$ consisting of open symmetric entourages such that for each $V_i \in \mathcal{B}_i$ and for each $x_i \in E_i$, $V_i[x_i]$ is Γ_i-convex. Let $\mathcal{S}$ be the family of all sets of the form $\{(x, y) \in E \times E : (x_i, y_i) \in V_i\}$ for $i \in I$ and $V_i \in \mathcal{B}_i$. It is easy to check that $\mathcal{S}$ is a subbase of $\mathcal{U}$. Let $\mathcal{B}$ be the base generated by $\mathcal{S}$, that is,

$$\mathcal{B} = \{V = V^1 \cap \cdots \cap V^n : V^i \in \mathcal{S},\ i = 1, \cdots, n;\ n \in \mathbb{N}\}.$$

Since each V^j is of the form

$$V^j = \{(x, y) \in E \times E : (x_{i_j}, y_{i_j}) \in V_{i_j}\}$$

for some $i_j \in I$ and $V_{i_j} \in \mathcal{B}_{i_j}$, we see that for each $x \in E$,

$$V^j[x] = \Pi_{i \in I \setminus \{i_j\}} E_i \times V_{i_j}[\pi_{i_j}(x)],$$

so that each $V^j[x]$ is Γ-convex by Lemma 4.1. Therefore,

$$V[x] = \Pi_{i \in I \setminus \{i_j : j=1,\cdots,n\}} E_i \times \Pi_{j=1}^n V_{i_j}[\pi_{i_j}(x)]$$

is Γ-convex. $\square$

Lemma 4.3. *Let $(E; \mathcal{U}, \Gamma) = (\Pi_{i \in I} E_i; \Pi_{i \in I} \mathcal{U}_i, \Pi_{i \in I} \Gamma_i)$ be the product locally G-convex uniform space of the family of locally G-convex uniform spaces $(E_i; \mathcal{U}_i, \Gamma_i)$, $i \in I$. Then $A := \Pi_{i \in I} A_i$ is almost Γ-convex in E provided for each $i \in I$, A_i is an almost Γ_i-convex subset of E_i.*

Proof. Let $\{x^1, \cdots, x^n\} \in \langle \Pi_{i \in I} A_i \rangle$ and $U \in \mathcal{U}$. We see from the proof of the proceeding lemma that there are $\{i_j : j = 1, \cdots, m\} \subseteq I$ and $V_{i_j} \in \mathcal{U}_{i_j}$, $j = 1, \cdots, m$, such that $V := \cap_{j=1}^m V^j \subseteq U$, where each $V^j = \{(x, y) \in E \times E : (x_{i_j}, y_{i_j}) \in V_{i_j}\}$. Put

$$U_i = \begin{cases} E_i \times E_i, & \text{if } i \in I \setminus \{i_j : j = 1, \cdots, m\}; \\ V_{i_j}, & \text{if } i \in \{i_j : j = 1, \cdots, m\}. \end{cases}$$

Then each $U_i \in \mathcal{U}_i$. Since $\{\pi_i x^1, \cdots, \pi_i x^n\} \in \langle A_i \rangle$ and A_i is almost Γ_i-convex, there exists $\{y_i^1, \cdots, y_i^n\} \in \langle A_i \rangle$ such that

$$y_i^k \in U_i[\pi_i x^k], \;\; \forall k = 1, \cdots, n, \text{ and}$$
$$\Gamma_i\text{-co}\{y_i^1, \cdots, y_i^n\} \subseteq A_i.$$

For each $k = 1, \cdots, n$, let $y^k = (y_i^k)_{i \in I} \in \Pi_{i \in I} E_i$. It is obvious that

$$\{y^1, \cdots, y^n\} \in \langle \Pi_{i \in I} A_i \rangle,$$

and $y^k \in V[x^k] \subseteq U[x^k]$ for any $k = 1, \cdots, n$. Furthermore, since both of Γ-co$\{y^1, \cdots, y^n\}$ and $\Pi_{i\in I}\Gamma_i$-co$\{y_i^1, \cdots, y_i^n\}$ contain $\{y^1, \cdots, y^n\}$ and since $\Pi_{i\in I}\Gamma_i$-co$\{y_i^1, \cdots, y_i^n\}$ is Γ-convex by Lemma 4.1, we infer that

$$\Gamma\text{-co}\{y^1, \cdots, y^n\} \subseteq \Pi_{i\in I}\Gamma_i\text{-co}\{y_i^1, \cdots, y_i^n\} \subseteq \Pi_{i \in I} A_i.$$

This shows that $\Pi_{i \in I} A_i$ is almost Γ-convex. □

Lemma 4.4. *Suppose for each $i \in I$ the locally G-convex uniform space $(E_i; \mathcal{U}_i, \Gamma_i)$ has the properties:*
(a) *$\Gamma_i(C_i)$ is closed and Γ_i-convex for any $C_i \in \langle E_i \rangle$, and*
(b) *there is a base $\mathcal{B}_i$ of $\mathcal{U}_i$ consisting of closed symmetric entourages such that for any $V_i \in \mathcal{B}_i$, $V_i[K_i]$ is Γ_i-convex whenever K_i is a Γ_i-convex subset of E_i.*
Then $(E; \mathcal{U}, \Gamma) = (\Pi_{i\in I} E_i; \Pi_{i\in I}\mathcal{U}_i, \Pi_{i\in I}\Gamma_i)$ has the properties:
(i) *Γ_C is closed and Γ-convex for any $C \in \langle E \rangle$, and*
(ii) *there is a base $\mathcal{B}$ of $\mathcal{U}$ consisting of closed symmetric entourages such that for any $V \in \mathcal{B}$, $V[K]$ is Γ-convex whenever K is a Γ-convex subset of E.*

Proof. Let $C \in \langle E \rangle$. Noting that Γ-co$(C) = \Gamma_C = \Pi_{i\in I}\Gamma_i(\pi_i(C))$ and for each $i \in I$, $\Gamma_i(\pi_i(C))$ is closed and Γ_i-convex, we infer that Γ_C is closed and Γ-convex. This establishes (i).

(ii) can be proved in a like manner as that of Proposition 4.2. □

Theorem 4.5. *Suppose for each $i \in I$ the locally G-convex uniform space*

$(E_i;\mathcal{U}_i,\Gamma_i)$ *has the properties:*

(a) $\Gamma_i(C_i)$ *is closed and* Γ_i*-convex for any* $C_i \in \langle E_i \rangle$*, and*

(b) *there is a base* $\mathcal{B}_i$ *of* $\mathcal{U}_i$ *consisting of closed symmetric entourages such that for any* $V_i \in \mathcal{B}_i$*,* $V_i[K_i]$ *is* Γ_i*-convex whenever* K_i *is a* Γ_i*-convex subset of* E_i*.*

Assume for each $i \in I$*,* A_i *is a dense almost* Γ_i*-convex subset of a compact subset* K_i *of* E_i*. Let* $K = \Pi\{K_i : i \in I\}$*,* $A_i' = \Pi\{A_\lambda : \lambda \in I, \lambda \neq i\}$ *and* $K_i' = \Pi\{K_\lambda : \lambda \in I, \lambda \neq i\}$*. If for each* $i \in I$*,* $F_i : K_i' \multimap K_i$ *is a closed multimap so that* $F_i(x_i')$ *is* Γ_i*-convex for each* $x_i' \in A_i'$*. Then* $\cap\{Gr(F_i) : i \in I\} \neq \varnothing$*.*

Proof. For each $i \in I$, define $H_i : K \to K_i$ by

$$H_i(x) = F_i(x_i'),$$

where x_i' is the projection of x on K_i'. Since H_i is the composition of a continuous function and an u.s.c. multimap, it is u.s.c., and so it has closed graph. Define $H : K \multimap K$ by $H(x) = \Pi_{i\in I} H_i(x)$ and let $A = \Pi_{i\in I} A_i$. We see from Lemma 4.3 that A is almost Γ-convex, and $H(x)$ is Γ-convex for any $x \in A$ by Lemma 4.1. Since each F_i has closed graph, so is each H_i, and hence H has closed graph. In addition, the hypotheses (a) and (b) and Lemma 4.4 show that the product locally G-convex uniform space $(E;\mathcal{U},\Gamma)$ posseses the properties (a) and (b) of Theorem 3.7. Consequently, all of the requirements of Theorem 3.7 are satisfied, and hence H has a fixed point $\hat{x}$. Obviously, $\hat{x} \in \cap_{i\in I} Gr(F_i)$. □

A real-valued function $\varphi : E \to \mathbb{R}$ on a G-convex space $(E;\Gamma)$ is said to quasi-convex if for each $\lambda \in \mathbb{R}$, the set $\{x \in E : \varphi(x) \leq \lambda\}$ is Γ-convex; and quasi-concave if for each $\lambda \in \mathbb{R}$, the set $\{x \in E : \varphi(x) \geq \lambda\}$ is Γ-convex.

Theorem 4.6. *Suppose for* $i = 1, 2$ *the locally* G*-convex uniform space* $(E_i;\mathcal{U}_i,\Gamma_i)$ *has the properties:*

(a) $\Gamma_i(C_i)$ *is closed and* Γ*-convex for any* $C_i \in \langle E_i \rangle$*, and*

(b) *there is a base* $\mathcal{B}_i$ *of* $\mathcal{U}_i$ *consisting of closed symmetric entourages such that for any* $V_i \in \mathcal{B}_i$*,* $V_i[K_i]$ *is* Γ_i*-convex whenever* K_i *is a* Γ_i*-convex subset of* E_i*.*

Let K_i *be a compact subset of the locally* G*-convex uniform space* $(E_i;\mathcal{U}_i,\Gamma_i)$*, and*

A_i be a dense almost Γ_i-convex subset of K_i, $i = 1, 2$. If $f : K_1 \times K_2 \to \mathbb{R}$ is continuous and satisfies that
(i) *for all $x \in A_1$, $f(x,)$ is quasi-convex on A_2;*
(ii) *for all $y \in A_2$, $f(, y)$ is quasi-concave on A_1,*
then $\max_{x\in K_1} \min_{y\in K_2} f(x, y) = \min_{y\in K_2} \max_{x\in K_1} f(x, y)$.

Proof. Since f is continuous on the compact set $K_1 \times K_2$, both of

$$\max_{x\in K_1} \min_{y\in K_2} f(x, y) \text{ and } \min_{y\in K_2} \max_{x\in K_1} f(x, y)$$

are well-defined. Now, let $\epsilon > 0$ be given. For any $\bar{y} \in K_2$, since

$$\max_{x\in K_1} f(x, \bar{y}) - \epsilon \geq \min_{y\in K_2} \max_{x\in K_1} f(x, y) - \epsilon,$$

there is $\bar{x} \in K_1$ such that $f(\bar{x}, \bar{y}) > \min_{y\in K_2} \max_{x\in K_1} f(x, y) - \epsilon$. In a like manner, for any $\bar{x} \in K_1$, there is $\bar{y} \in K_2$ such that $f(\bar{x}, \bar{y}) < \max_{x\in K_1} \min_{y\in K_2} f(x, y) + \epsilon$. So we can define $F_1 : K_2 \multimap K_1$ and $F_2 : K_1 \multimap K_2$ by

$$F_1(y) = \{x \in K_1 : f(x, y) \geq \min_{y\in K_2} \max_{x\in K_1} f(x, y) - \epsilon\}$$
$$F_2(x) = \{y \in K_2 : f(x, y) \leq \max_{x\in K_1} \min_{y\in K_2} f(x, y) + \epsilon\}.$$

By the continuity of f on $K_1 \times K_2$, we see that both of F_1 and F_2 are closed multimaps. Furthermore, by assumptions (i) and (ii), $F_1(y)$ is Γ_1-convex for all $y \in A_2$, and $F_2(x)$ is Γ_2-convex for all $x \in A_1$. Thus, it follows from Theorem 4.5 that $Gr(F_1) \cap Gr(F_2) \neq \varnothing$, that is, there exists $(x_\epsilon, y_\epsilon) \in K_1 \times K_2$ such that

$$\min_{y\in K_2} \max_{x\in K_1} f(x, y) - \epsilon \leq f(x_\epsilon, y_\epsilon) \leq \max_{x\in K_1} \min_{y\in K_2} f(x, y) + \epsilon.$$

Then,

$$\begin{aligned} \liminf_{\epsilon\to 0} \left(\min_{y\in K_2} \max_{x\in K_1} f(x, y) - \epsilon \right) &\leq \liminf_{\epsilon\to 0} f(x_\epsilon, y_\epsilon) \\ &\leq \limsup_{\epsilon\to 0} f(x_\epsilon, y_\epsilon) \\ &\leq \limsup_{\epsilon\to 0} \left(\max_{x\in K_1} \min_{y\in K_2} f(x, y) + \epsilon \right), \end{aligned}$$

and hence $\min_{y\in K_2}\max_{x\in K_1} f(x,y) \le \max_{x\in K_1}\min_{y\in K_2} f(x,y)$. But, it is obvious that $\max_{x\in K_1}\min_{y\in K_2} f(x,y) \le \min_{y\in K_2}\max_{x\in K_1} f(x,y)$. Therefore,

$$\max_{x\in K_1}\min_{y\in K_2} f(x,y) = \min_{y\in K_2}\max_{x\in K_1} f(x,y).$$

□

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