Fixed Point Theorems for Condensing Maps in S-KKM Class

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This paper presents some new fixed point results for condensing multimaps in s-KKM class in the setting of locally G-convex uniform spaces. We mainly show that every l.s.c., closed and generalized condensing self-multimap with s-KKM property on a complete locally G-convex space has a fixed point. Some applications to quasi-equilibrium problem of these fixed point results are also given.

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1. Introduction and Preliminaries

Chang et al. [3] introduced the S-KKM class, where a lot of interesting fixed point theorems on locally convex topological vector spaces were established. Later, these results were extended to locally G-convex uniform spaces in [8]. On the other hand, we introduced the concepts of measure of precompactness and condensing multimaps on locally G-convex uniform spaces in [5]. The intent of this paper is to present some fixed point theorems for condensing multimaps in S-KKM class in the setting of locally G-convex uniform spaces. These results partially cover our previous results in [3], [4] and [8].

We now recall some basic definitions and facts. For a nonempty set Y, 2^Y denotes the class of all subsets of Y and $\langle Y \rangle$ denotes the class of all nonempty finite subsets of Y. A multimap $T : X \to 2^Y$ is a function from a set X into the power set 2^Y of Y. The notation $T : X \multimap Y$ stands for a multimap $T : X \to 2^Y$ having nonempty values.

For a multimap $T : X \to 2^Y$, $A \subseteq X$ and $B \subseteq Y$, the image of A under T is the set $T(A) = \bigcup_{x \in A} T(x)$; and the inverse image of B under T is $T^-(B) = \{x \in X : T(x) \cap B \neq \varnothing\}$.

All topological spaces are supposed to be Hausdorff. The closure of a subset X of a topological space is denoted by $\overline{X}$. Let X and Y be two topological spaces. A multimap $T : X \to 2^Y$ is said to be

(a) lower semicontinuous(l.s.c.) if $T^-(B)$ is open in X for each open subset B of Y;

(b) compact if $T(X)$ is contained in a compact subset of Y;

(c) closed if its graph $Gr(T) = \{(x, y) : y \in T(x), x \in X\}$ is a closed subset of $X \times Y$.

For $n \geq 0$, Δ_n denotes the standard n-simplex of $\mathbb{R}^{n+1}$, that is,

$$\Delta_n = \left\{ \alpha = (\alpha_0, \cdots, \alpha_n) \in \mathbb{R}^{n+1} : \alpha_i \geq 0 \text{ for all } i \text{ and } \sum_{i=0}^{n} \alpha_i = 1 \right\};$$

and $\{\mathbf{e}_0, \cdots, \mathbf{e}_n\}$, the standard basis of $\mathbb{R}^{n+1}$, is the set of the vertices of Δ_n.

Definition 1.1.(Park and Kim [10]) *A generalized convex space or a G-convex space $(E; \Gamma)$ consists of a topological space E and a map $\Gamma : \langle E \rangle \multimap E$ such that*

(a) *for any $A, B \in \langle E \rangle$, $A \subseteq B$ implies $\Gamma(A) \subseteq \Gamma(B)$; and*

(b) *for each $A = \{a_0, \cdots, a_n\} \in \langle E \rangle$ with $|A| = n + 1$, there exists a continuous* *function* $\varphi_A : \Delta_n \to \Gamma(A)$ *such that if* $0 \le i_0 < i_1 < \cdots < i_k \le n$*, then* $\varphi_A(co\{\mathbf{e}_{i_0}, \cdots, \mathbf{e}_{i_k}\}) \subseteq \Gamma(\{a_{i_0}, \cdots, a_{i_k}\})$.

In this paper, we assume that a G-convex space $(E;\Gamma)$ *always satisfies the extra condition:* $x \in \Gamma(\{x\})$ *for any* $x \in E$.

A subset K *of a G-convex space* $(E;\Gamma)$ *is said to be* Γ*-convex if for any* $A \in \langle K \rangle$, $\Gamma(A) \subseteq K$*. For a nonempty subset* Q *of* E*, the* Γ*-convex hull of* Q*, denoted by* Γ*-co(Q), is defined by*

$$\Gamma\text{-}co(Q) = \cap\{C : Q \subset C \subset E, C \text{ is } \Gamma\text{-convex}\}.$$

It is easy to see that Γ-co(Q) is the smallest Γ-convex subset of E containing Q. For convenience, we also express $\Gamma(A)$ by Γ_A. By the definition of a G-convex space $(E;\Gamma)$, it is easy to check that $A \subseteq \Gamma_A$ for any $A \in \langle E \rangle$. If X is a nonempty Γ-convex subset of E, then $(X;\Gamma|_{\langle X \rangle})$ is also a G-convex space.

The concept of S-KKM property in G-convex spaces was introduced in [8]:

Definition 1.2. *Let* X *be a nonempty set,* $(E;\Gamma)$ *a G-convex space and* Y *a topological space. If* $S : X \multimap E$, $T : E \multimap Y$ *and* $F : X \multimap Y$ *are three multimaps satisfying*

$$T(\Gamma_{S(A)}) \subseteq F(A)$$

for any $A \in \langle X \rangle$*, then* F *is called a S-KKM mapping with respect to* T*. A multimap* $T : E \multimap Y$ *is said to have the S-KKM property if for any S-KKM mapping* F *with respect to* T*, the family* $\{\overline{F(x)} : x \in X\}$ *has the finite intersection property. The class S-KKM*(X, E, Y) *is defined to be the set* $\{T : E \multimap Y : T$ *has the S-KKM property*$\}$.

In the case that $X = E$ and S is the identity mapping 1_X, S-KKM(X, E, Y) is abbreviated as KKM(E, Y), and a 1_X-KKM mapping with respect to T is called a KKM mapping with respect to T, and 1_X-KKM property is called KKM property. For any nonempty set X, G-convex space $(E;\Gamma)$, topological space Y and any $S : X \multimap E$, one has KKM$(E, Y) \subseteq S$-KKM(X, E, Y).

A uniformity for a set X is a nonempty family $\mathcal{U}$ of subsets of $X \times X$ such that
(a) each member of $\mathcal{U}$ contains the diagonal Δ;

(b) if $U \in \mathcal{U}$, then $U^{-1} \in \mathcal{U}$;
(c) if $U \in \mathcal{U}$, then $V \circ V \subseteq U$ for some V in $\mathcal{U}$;
(d) if U and V are members of $\mathcal{U}$, then $U \cap V \in \mathcal{U}$; and
(e) if $U \in \mathcal{U}$ and $U \subseteq V \subseteq X \times X$, then $V \in \mathcal{U}$.

Every member V in $\mathcal{U}$ is called an entourage. An entourage V is said to symmetric if $(x, y) \in V$ whenever $(y, x) \in V$.

If $(X,\mathcal{U})$ is a uniform space, then the topology $\mathcal{T}$ induced by $\mathcal{U}$ is the family of all subsets W of X such that for each x in W there is U in $\mathcal{U}$ such that $U[x] \subseteq W$, where $U[x]$ is defined as $\{y \in X : (x, y) \in U\}$. If H is a subset of X and U is in $\mathcal{U}$, then $U[H] := \cup_{x \in H} U[x]$. For details of uniform spaces we refer to [6].

Definition 1.3.(Watson [12]) *A G-convex uniform space $(E;\mathcal{U}, \Gamma)$ is a G-convex space so that its topology is induced by a uniformity $\mathcal{U}$. A G-convex uniform space $(E;\mathcal{U}, \Gamma)$ is said to be a locally G-convex uniform space if the uniformity $\mathcal{U}$ has a base $\mathcal{B}$ consisting of open symmetric entourages such that for each $V \in \mathcal{B}$*
(1.3.1) *$V[x]$ is Γ-convex for any $x \in E$; and*
(1.3.2) *$V[K]$ is Γ-convex whenever K is a Γ-convex subset of X.*

A subset S of a uniform space E is said to be precompact if, for any entourage V, there is a finite subset N of E such that $S \subseteq V[N]$. In this paper, **for a locally G-convex uniform space $(E;\mathcal{U}, \Gamma)$, the convex structure Γ is assumed to have the property that Γ-co(A) is precompact whenever A is precompact, and $\mathcal{B}$ will always denotes a base of $\mathcal{U}$ so that it has the properties described in Definition 1.3.**

Finally, we state a fixed point theorem of Kuo et al. [8, Theorem 3.9] which will be quoted in the sequel.

Theorem 1.4. *Let X be any nonempty set, $(E;\mathcal{U}, \Gamma)$ a locally G-convex uniform space and $s : X \to E$. If $T \in s$-KKM(X, E, E) satisfies that*
(1.4.1) *T is compact and closed;*
(1.4.2) *$\overline{T(E)} \subseteq s(X)$, then T has a fixed point.*

If s is surjective, then condition (1.4.2) is surely satisfied, so Theorem 1.4 holds for a surjection s.

2. Fixed Point Theorems

We begin this section by presenting a fixed point theorem of Mönch type, cf. Agarwal and O'Regan [1] , [2] and the references therein.

Theorem 2.1. *Let X be a nonempty closed Γ-convex subset of a complete locally G-convex uniform space $(E;\mathcal{U},\Gamma)$ and $s : X \to E$ a surjection. Assume $T \in s\text{-}KKM(X,E,E)$ is closed and satisfies the following conditions:*
(2.1.1) *T maps compact sets into precompact sets;*
(2.1.2) *there is $x_0 \in X$ such that if $C \subseteq X$ is countable and $C \subseteq \overline{\Gamma\text{-}co}(T(\overline{C}) \cup \{x_0\})$, then C is precompact;*
(2.1.3) *for any precompact subset A of X there is a countable subset B of A with $\overline{B} = \overline{A}$;*
(2.1.4) *$T(\overline{A}) \subseteq \overline{T(A)}$ for any precompact subset A of X.*
Then T has a fixed point.

Proof. Let $D_0 = \{x_0\}$ and $D_n = \Gamma\text{-co}(T(D_{n-1}) \cup \{x_0\})$ for $n \in \mathbb{N}$, and $D = \bigcup_{n=0}^{+\infty} D_n$. Noting that each D_n is Γ-convex and

$$D_0 \subseteq D_1 \subseteq \cdots \subseteq D_n \cdots \subseteq D \subseteq X,$$

we deduce that D is Γ-convex. Also, it is obvious that

$$\begin{aligned} D &= \bigcup_{n=1}^{+\infty} \Gamma\text{-co}\left(T(D_{n-1}) \cup \{x_0\}\right) \\ &\subseteq \Gamma\text{-co}\left(T(D) \cup \{x_0\}\right). \end{aligned} \tag{2.1}$$

On the other hand, since $\{D_n\}_0^{+\infty}$ is increasing and since

$$\Gamma\text{-co}\left(T(D) \cup \{x_0\}\right) = \bigcup \left\{\Gamma\text{-co}(A) : A \in \langle T(D) \cup \{x_0\}\rangle\right\}, \tag{2.2}$$

cf. [11, Lemma 2.1], we have that for any $A \in \langle T(D) \cup \{x_0\}\rangle$ there is a D_k with

$A \in \langle T(D_{k-1}) \cup \{x_0\}\rangle$, so

$$\begin{aligned}\Gamma\text{-co}(A) &\subseteq \Gamma\text{-co}(T(D_{k-1}) \cup \{x_0\}) \\ &\subseteq \bigcup_{n=1}^{+\infty} \Gamma\text{-co}(T(D_{n-1}) \cup \{x_0\}) = D. \end{aligned} \tag{2.3}$$

It follows from (2.1), (2.2) and (2.3) that

$$D = \Gamma\text{-co}(T(D) \cup \{x_0\}). \tag{2.4}$$

We now show by induction that each D_n is precompact. Obviously, D_0 is precompact. Assume D_k is precompact for $k \geq 1$. Condition (2.1.1) guarantees that $T(\overline{D_k})$ is precompact, so $\overline{T(\overline{D_k})} \cup \{x_0\}$ is compact. By our hypotheses on locally G-convex uniform spaces, $\Gamma\text{-co}(\overline{T(\overline{D_k})} \cup \{x_0\})$ is precompact. Thus D_{k+1} is precompact once we note that

$$D_{k+1} = \Gamma\text{-co}(T(D_k) \cup \{x_0\}) \subseteq \Gamma\text{-co}(\overline{T(\overline{D_k})} \cup \{x_0\}).$$

This completes the induction for showing each D_n is precompact.

Next, condition (2.1.3) implies that for each $n \in \mathbb{N} \cup \{0\}$, there is a countable subset C_n of D_n such that $\overline{C_n} = \overline{D_n}$. Let $C = \bigcup_{n=0}^{+\infty} C_n$. Since

$$\bigcup_{n=0}^{+\infty} D_n \subseteq \bigcup_{n=0}^{+\infty} \overline{D_n} \subseteq \overline{\bigcup_{n=0}^{+\infty} D_n},$$

we have

$$\overline{\bigcup_{n=0}^{+\infty} \overline{D_n}} = \overline{\bigcup_{n=0}^{+\infty} D_n} = \overline{D}$$

and

$$\overline{\bigcup_{n=0}^{+\infty} \overline{D_n}} = \overline{\bigcup_{n=0}^{+\infty} \overline{C_n}} = \overline{\bigcup_{n=0}^{+\infty} C_n} = \overline{C}.$$

Consequently,

$$\overline{C} = \overline{D}. \tag{2.5}$$

Combining (2.4) and (2.5) yields that

$$\begin{aligned} C \subseteq \overline{C} = \overline{D} &= \overline{\Gamma\text{-co}}(T(D) \cup \{x_0\}) \\ &\subseteq \overline{\Gamma\text{-co}}(T(\overline{D}) \cup \{x_0\}) \\ &= \overline{\Gamma\text{-co}}(T(\overline{C}) \cup \{x_0\}). \end{aligned} \tag{2.6}$$

Since C is countable, it follows from (2.1.2) and (2.7) that $\overline{C}$ is compact, and hence $\overline{D}$ is compact by (2.5).

Finally, notice that (2.4) implies $T(D) \subseteq D$, which together with (2.1.4) shows $T(\overline{D}) \subseteq \overline{D}$. Putting $K = s^{-}(\overline{D})$, we see that $s : K \to \overline{D}$ is surjective and $T \in s\text{-KKM}(K, \overline{D}, \overline{D})$ is compact and closed. The existence of a fixed point for T now follows from Theorem 1.4. □

That a multimap $T : X \multimap Y$ is l.s.c. could be phrased as: *For any $x \in X$ and any net $\{x^\alpha\}$ in X which converges to x and any $y \in T(x)$, there exists a net $\{y^\alpha\}$ in Y such that for each α, $y^\alpha \in T(x^\alpha)$ and y^α converges to y.* By means of this characterization, it is easy to see that $T(\overline{A}) \subseteq \overline{T(A)}$ for any subset A of X. Hence, the above theorem holds true when (2.1.4) is replaced with
(2.1.4)′ T is l.s.c.

Theorem 2.2. *X be a nonempty closed Γ-convex subset of a complete locally G-convex uniform space $(E; \mathcal{U}, \Gamma)$ and $s : X \to E$ a surjection. Assume $T \in$ s-KKM(X, X, X) is closed and satisfies*
(2.2.1) *there is $x_0 \in X$ such that if $A \subseteq X$ with $A \subseteq \Gamma\text{-}co(T(A) \cup \{x_0\})$, then A is precompact;*
(2.2.2) *for any precompact Γ-convex subset A of X with $T(x) \cap \overline{A} \neq \varnothing$ for each $x \in \overline{A}$, the multimap $H : \overline{A} \multimap \overline{A}$ defined by $H(x) = T(x) \cap \overline{A}$ is in s-KKM$(s^{-}(\overline{A}), \overline{A}, \overline{A})$.*
Then T has a fixed point.

Proof. Let D and D_n, $n \in \mathbb{N} \cup \{0\}$, be as in the proof of Theorem 2.1. We have seen that D is Γ-convex and (2.4) holds. Thus, by (2.2.1) and (2.4) , D is precompact. Moreover, it follows from (2.4) that $T(D) \subseteq D$. Hence, if $x \in D$, we surely have

$$T(x) \cap \overline{D} \supseteq T(x) \cap D = T(x) \neq \varnothing.$$

If $x \in \overline{D} \setminus D$, choose a net $\{x_\alpha\}$ in D such that $x_\alpha \to x$. For any α, choose $y_\alpha \in T(x_\alpha)$. Since $\{y_\alpha\} \subseteq T(D) \subseteq \overline{D}$ and $\overline{D}$ is compact, y_α has a subnet y_{α_j} so that $y_{\alpha_j} \to y$ for some $y \in \overline{D}$. Therefore the multimap $H : \overline{D} \multimap \overline{D}$ defined by $H(x) = T(x) \cap \overline{D}$ is well-defined. Let $K = s^{-}(\overline{D})$. Then $s : K \to \overline{D}$ is surjective.

Furthermore, H is in s-KKM$(K, \overline{D}, \overline{D})$ by (2.2.2). Clearly, H is compact and closed, so it has a fixed point by Theorem 1.4. $\square$

In the remainder of this section we investigate the fixed point problem for condensing multimaps in s-KKM class. At first, we quote some definitions and results in [5].

Definition 2.3.(Huang et al. [5]) *For a subset A of locally G-convex uniform space $(E;\mathcal{U},\Gamma)$, the measure of the precompactness of A is defined to be*

$$\Psi(A) = \{V \in \mathcal{B} : A \subseteq V[S] \text{ for some precompact subset } S \text{ of } E\}.$$

Clearly, the larger $\Psi(A)$ the more nearly is A precompact. In fact, we have

Proposition 2.4.(Huang et al. [5]) *Let A and B be subsets of $(E;\mathcal{U},\Gamma)$. Then*
(a) *A is precompact iff $\Psi(A) = \mathcal{B}$;*
(b) *$\Psi(A) \supseteq \Psi(B)$ if $A \subseteq B$;*
(c) *$\Psi(\Gamma\text{-}co(A)) = \Psi(A)$;*
(d) *$\Psi(A \cup B) = \Psi(A) \cap \Psi(B)$.*

Definition 2.5.(Huang et al. [5]) *Suppose X is a nonempty subset of a locally G-convex uniform space $(E;\mathcal{U},\Gamma)$ and Ψ is the measure of precompactness in Definition 2.3. A multimap $T : X \to 2^E$ is called condensing provided $\Psi(A) \subsetneqq \Psi(T(A))$ for any subset A that is not precompact. T is called generalized condensing if whenever $A \subseteq X$, $T(A) \subseteq A$ and $A \setminus \Gamma\text{-}co(T(A))$ is precompact, then A is precompact.*

It is obvious that every compact map or every map defined on a compact set is condensing. Also, every condensing map is generalized condensing.

Lemma 2.6.(Huang et al. [5]) *Let X be a nonempty Γ-convex subset of a locally G-convex uniform space $(E;\mathcal{U},\Gamma)$ and $T : X \multimap X$. Then,*
(a) *for any $x_0 \in X$, there is a precompact Γ-convex subset K of X such that*

$$T(K) \subseteq K \text{ and } K = \Gamma\text{-}co(T(K) \cup \{x_0\}).$$

provided that T is condensing;
(b) *there is a nonempty precompact Γ-convex subset K of X such that $T(K) \subseteq K$*

and

$$T(x) \cap \overline{K} \neq \varnothing \text{ for any } x \in \overline{K}.$$

provided that X is complete and T is generalized condensing and closed.

We now in a position to present fixed point results for condensing multimaps.

Theorem 2.7. *Let X be a nonempty closed Γ-convex subset of a complete locally G-convex uniform space $(E;\mathcal{U},\Gamma)$ and $s : X \to X$ a surjection. If $T \in s$-KKM(X, X, X) is closed, generalized condensing and satisfies that*

$$T(\overline{A}) \subseteq \overline{T(A)}$$

for any precompact subset A of X, then T has a fixed point.

Proof. By Lemma 2.6, there exists a precompact Γ-convex subset K of X such that $T(K) \subseteq K$, which in conjunction with $T(\overline{K}) \subseteq \overline{T(K)}$ shows that $T(\overline{K}) \subseteq \overline{K}$. Putting $D = s^{-}(\overline{K})$. Then $s : D \to \overline{K}$ is surjective and $T \in s$-KKM$(D, \overline{K}, \overline{K})$ is compact and closed. So, the conclusion follows from Theorem 1.4. □

Theorem 2.8. *Let X be a nonempty closed Γ-convex subset of a complete locally G-convex uniform space $(E;\mathcal{U},\Gamma)$ and $s : X \to X$ a surjection. If $T \in s$-KKM(X, X, X) is l.s.c., closed and generalized condensing, then T has a fixed point.*

Proof. Since T is l.s.c., we have that $T(\overline{A}) \subseteq \overline{T(A)}$ for any subset A of X, so the conclusion follows from Theorem 2.7. □

When X is a nonempty closed convex subset of a locally convex topological vector space, it is shown in [4] that every closed and generalized condensing self multimap on X with s-KKM property has a fixed point. Hence, it is interesting to ask whether the condition of lower semicontinuity can be dropped in Theorem 2.8.

Theorem 2.9. *Let X be a nonempty closed Γ-convex subset of a complete locally*

G-convex uniform space $(E;\mathcal{U},\Gamma)$ and $s : X \to X$ a surjection. Assume $T : X \multimap X$ satisfies

(2.9.1) *T is closed and generalized condensing;*

(2.9.2) *for any precompact subset A of X, if*

$$H(x) = T(x) \cap \overline{A} \neq \varnothing, \ \textit{for any } x \in \overline{A},$$

then $H \in s$-KKM$(s^{-}(\overline{A}), \overline{A}, \overline{A})$.

Then T has a fixed point.

Proof. Condition (2.9.1) and Lemma 2.6 show that there is a precompact Γ-convex subset K of X such that $T(K) \subseteq K$ and $T(x) \cap \overline{K} \neq \varnothing$ for any $x \in \overline{K}$. Define $H : \overline{K} \multimap \overline{K}$ by $H(x) = T(x) \cap \overline{K}$ for $x \in \overline{K}$. Then condition (2.9.2) gives us that $H \in s$-KKM$(s^{-}(\overline{K}), \overline{K}, \overline{K})$. Obviously, H is compact and closed. As H satisfies all requirements of Theorem 1.4, it has a fixed point $\hat{x}$ which is also a fixed point of T. □

If $T(\overline{K}) \subseteq \overline{T(K)}$ and $T \in s$-KKM(X, X, X), then $T \in s$-KKM$(s^{-}(\overline{K}), \overline{K}, \overline{K})$, so Theorem 2.7 can be derived from Theorem 2.9.

3. Applications

In this section, we shall deduce some quasi-equilibrium theorems as applications of the previous fixed point results.

Theorem 3.1. *Let X be a nonempty closed Γ-convex subset of a complete locally G-convex uniform space $(E;\mathcal{U},\Gamma)$, $s : X \to X$ a surjection, $f : X \times X \to \mathbb{R}$ an upper semicontinuous function and $H : X \multimap X$ a closed, condensing multimap with compact values. Assume the following two conditions holds:*

(3.1.1) *The function $M : X \to \mathbb{R}$ defined by*

$$M(x) = \max_{y \in H(x)} f(x, y)$$

is lower semicontinuous.

(3.1.2.) *The multimap $T : X \multimap X$ defined by*

$$T(x) = \{y \in H(x) : f(x, y) = M(x)\}$$

is in s-KKM(X,X,X) and satisfies $T(\overline{A}) \subseteq \overline{T(A)}$ for any precompact subset A of X.

Then there exists an $\hat{x} \in X$ such that $\hat{x} \in H(\hat{x})$ and $f(\hat{x},\hat{x}) = M(\hat{x})$.

Proof. Since f is u.s.c. and $H(x)$ is compact, $T(x)$ is nonempty for any $x \in X$. We claim that T is closed. Let $\{(x_\alpha, y_\alpha)\}$ be a net in $Gr(T)$ and $(x_\alpha, y_\alpha) \to (x,y)$. We have

$$\begin{aligned} f(x,y) \geq \limsup_\alpha f(x_\alpha, y_\alpha) &= \limsup_\alpha M(x_\alpha) \\ &\geq \liminf_\alpha M(x_\alpha) \\ &\geq M(x), \end{aligned}$$

where the last inequality follows from (3.1.1). Since H is closed and $y_\alpha \in H(x_\alpha)$ for any α, we see that $(x,y) \in Gr(H)$. Hence $y \in \{z \in H(x) : f(x,z) = M(x)\}$, which shows T is closed. In addition, T is condensing. In fact, if $A \subseteq X$ is not precompact, then, since H is condensing, we have from Proposition 2.4 that

$$\Psi(A) \subsetneqq \Psi(H(A)) \subseteq \Psi(T(A)),$$

This shows that T is condensing. Consequently, T has a fixed point $\hat{x}$ by Theorem 2.7, that is, $\hat{x} \in H(\hat{x})$ and $f(\hat{x},\hat{x}) = M(\hat{x})$. □

Suppose $\{(E_i; \mathcal{U}_i, \Gamma_i)\}_{i\in I}$ is any family of locally G-convex uniform spaces. Let $E = \Pi_{i\in I} E_i$ be equipped with product topology induced by the product uniformity $\mathcal{U} = \Pi_{i\in I}\mathcal{U}_i$, and the product convexity structure $\Gamma = \Pi_{i\in I}\Gamma_i : \langle E\rangle \multimap E$ defined by

$$\Gamma(A) = \Pi_{i\in I}\Gamma_i(\pi_i(A))$$

for each $A \in \langle E\rangle$, where $\pi_i : E \to E_i$ is the projection of E onto E_i. Then $(E; \mathcal{U}, \Gamma)$ is a locally G-convex uniform space, cf. [5]. Moreover, if each $(E_i; \mathcal{U}_i, \Gamma_i)$ is complete and if each Γ_i has the property that Γ_i-co(A_i) is precompact for any precompact subset A_i of E_i, then Γ has the similar property, that is, Γ-co(A) is precompact whenever A is precompact in E. To see this, let A be a precompact subset of E. Since each π_i is uniformly continuous, we see that each $\pi_i(A)$ is precompact in E_i. Now, noting that

$$A \subseteq \Pi_{i\in I}\Gamma_i\text{-co}(\pi_i(A)) \subseteq \overline{\Pi_{i\in I}\Gamma_i\text{-co}(\pi_i(A))} \subseteq \Pi_{i\in I}\overline{\Gamma_i\text{-co}}(\pi_i(A))$$

and $\Pi_{i\in I}\overline{\Gamma_i\text{-co}}(\pi_i(A))$ is Γ-convex, cf. [5, Lemma 2.5], we obtain that $\Gamma\text{-co}(A) \subseteq \Pi_{i\in I}\overline{\Gamma_i\text{-co}}(\pi_i(A))$. So, the compactness of $\Pi_{i\in I}\overline{\Gamma_i\text{-co}}(\pi_i(A))$ implies that $\Gamma\text{-co}(A)$ is precompact.

Lemma 3.2. *Let $\{X_i\}_{i\in I}$ be a family of nonempty closed Γ_i-convex subsets, each in a complete locally G-convex uniform space $(E_i;\mathcal{U}_i,\Gamma_i)$, K_i a nonempty compact subset of X_i and $T_i : X = \Pi_{i\in I}X_i \multimap K_i$ a l.s.c., closed multimap. Assume $T : X \multimap K = \Pi_{i\in I}K_i$ defined by $T(x) = \Pi_{i\in I}T_i(x)$ for each $x \in X$ has the KKM property. Then there is $\hat{x} \in X$ such that $\hat{x}_i \in T_i(\hat{x})$, for any $i \in I$.*

Proof. Obviously, T is a compact and closed multimap. Furthermore, since each T_i is l.s.c., so is T, cf. [7, Theorem 7.3.12]. Therefore, T has a fixed point $\hat{x}$ by Theorem 2.8, that is, $\hat{x}_i \in T_i(x)$ for any $i \in I$. □

Theorem 3.3. *Let $\{X_i\}_{i\in I}$ be a family of nonempty closed Γ_i-convex subsets, each in a complete locally G-convex uniform space $(E_i;\mathcal{U}_i,\Gamma_i)$, K_i a nonempty compact subset of X_i, $H_i : X = \Pi_{i\in I}X_i \multimap K_i$ a closed multimap with compact values, and f_i, $g_i : X^{-i}\times X_i \to \mathbb{R}$ be upper semicontinuous functions, where X^{-i} denotes $\Pi_{j\in I\setminus\{i\}}X_j$. Assume that*

(3.3.1) *$g_i(x) \leq f_i(x)$ for any $x \in X$;*

(3.3.2) *for any $i \in I$, the function $M_i : X = X^{-i} \times X_i \to \mathbb{R}$ defined by*

$$M_i(x) = \max_{y\in H_i(x)} g_i(x^{-i}, y)$$

is lower semicontinuous; and

(3.3.3) *for any $i \in I$, the multimap $T_i : X \multimap K_i$ defined by*

$$T_i(x) = \{y \in H_i(x) : f_i(x^{-i}, y) \geq M_i(x)\}$$

is l.s.c.

(3.3.4) *$T : X \multimap K = \Pi_{i\in I}K_i$ defined by $T(x) = \Pi_{i\in I}T_i(x)$ for each $x \in X$ has the KKM property.*

Then there exists an $\hat{x} \in X$ such that for each $i \in I$,

$$\hat{x}_i \in H_i(\hat{x}) \text{ and } f_i(\hat{x}^{-i}, \hat{x}_i) \geq M_i(\hat{x}).$$

Proof. Firstly, note each $T_i(x)$ is nonempty by (3.3.1) since each $H_i(x)$ is compact and $g_i(x^{-i},\ )$ is u.s.c. on $H_i(x)$. Next, just as in the proof of Theorem 3.1, $Gr(T_i)$ is closed in $X \times K_i$ and T_i is compact. Applying Lemma 3.2, we infer that there is an $\hat{x} \in X$ such that $\hat{x}_i \in T_i(x)$ for any $i \in I$, that is, $\hat{x}_i \in H_i(\hat{x})$ and $f_i(\hat{x}^{-i}, \hat{x}_i) \geq M_i(\hat{x})$. □

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