Coincidence and fixed points on product generalized convex spaces

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In this paper we at first establish some new results of coincidence and fixed points for a family of multimaps on a product G-convex space, and then apply them to the study of system of inequalities, minimax problem, maximal elements and equilibrium points of abstract economy.

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1. Introduction and Preliminaries

Recently, the theory of coincidence and fixed points for a family of multimaps on a product space of topological vector spaces has been investigated by many authors, cf.[1], [2], [7], [8], [12] and the references therein. In this paper we shall make a study of this type of problems on a product G-convex space instead of a product topological vector space. We at first establish some new results of coincidence and fixed points for a family of multimaps on a product G-convex space, and then apply them to the study of system of inequalities, minimax problem, maximal elements and equilibrium points of abstract economy.

We begin with recalling some notations and terminology concerned with multimaps which will be used throughout the paper.

For a nonempty set Y, 2^Y and $\langle Y \rangle$ denote the class of all subsets of Y and the class of all nonempty finite subsets of Y respectively. For $A \in \langle Y \rangle$, $|A|$ denotes the cardinality of A. A multimap $T : X \to 2^Y$ is a function from a set X into the power set 2^Y of Y. The notation $T : X \multimap Y$ stands for a multimap $T : X \to 2^Y$ having nonempty values. If $B \subseteq Y$, then the inverse image of B under T is $T^{-}(B) = \{x \in X : T(x) \cap B \neq \varnothing\}$.

All topological spaces are supposed to be Hausdorff. The closure of a subset A of a topological space is denoted by $\mathrm{cl}(A)$. A subset A of a topological space X is compactly open if for every nonempty compact subset K of X, $A \cap K$ is open in K. The compactly interior of A is defined by

$$\mathrm{cint}(A) = \cup\{B \subseteq X : B \subseteq A \text{ and } B \text{ is compactly open in } X\},$$

cf.[3].

Definition 1.1.[13]. *Let X and Y be two topological spaces. A multimap $T : X \to 2^Y$ is said to be transfer compactly open valued if for any $x \in X$ and $y \in T(x)$ there is an $x' \in X$ such that $y \in \mathrm{cint}(T(x'))$. A function $f : X \times Y \to \mathbb{R}$ is said to be transfer compactly l.s.c. (* resp. *u.s.c.) on Y if for each $y \in Y$ and each $\lambda \in \mathbb{R}$ with $y \in \{z \in Y : f(x, z) > \lambda\}$ (* resp. *$y \in \{z \in Y : f(x, z) < \lambda\}$) there exists an $x' \in X$ such that*

$$y \in \mathrm{cint}(\{z \in Y : f(x', z) > \lambda\})$$

$$(\text{resp. } y \in \mathrm{cint}(\{z \in Y : f(x', z) < \lambda\})).$$

The following lemmas are well-known.

Lemma 1.2.[8] *Let X and Y be two topological spaces and $f : X \times Y \to \mathbb{R}$. Then f is transfer compactly l.s.c.* (resp. *u.s.c.*) *on Y if and only if for any $\lambda \in \mathbb{R}$, the multimap $F : X \to 2^Y$ defined by*

$$F(x) = \{y \in Y : f(x, y) > \lambda\}$$

$$(\text{resp. } F(x) = \{y \in Y : f(x, y) < \lambda\})$$

is transfer compactly open valued.

Lemma 1.3.[8] *Let X and Y be two topological spaces and $T : X \to 2^Y$. Then the following two statements are equivalent.*
(1.3.1) *For each $x \in X$, $T(x)$ is nonempty and $T^- : Y \to 2^X$ is transfer compactly open valued.*
(1.3.2) $X = \cup_{y\in Y} \mathrm{cint}(T^-(y))$.

For $n \geq 0$, Δ_n denotes the standard n-simplex of $\mathbb{R}^{n+1}$, that is,

$$\Delta_n = \left\{ \alpha = (\alpha_0, \cdots, \alpha_n) \in \mathbb{R}^{n+1} : \alpha_i \geq 0 \text{ for all } i \text{ and} \sum_{i=0}^{n} \alpha_i = 1 \right\};$$

and $\{\mathbf{e}_0, \cdots, \mathbf{e}_n\}$, the standard basis of $\mathbb{R}^{n+1}$, is the set of the vertices of Δ_n.

The notion of a generalized convex space was first introduced by Park and Kim [10]:

Definition 1.4. *A generalized convex space or a G-convex space $(E; \Gamma)$ consists of a topological space E and a map $\Gamma : \langle E\rangle \multimap E$ such that*
(1.4.1) *for any $A, B \in \langle E\rangle$, $A \subseteq B$ implies $\Gamma(A) \subseteq \Gamma(B)$; and*
(1.4.2) *for each $A = \{a_0, \cdots, a_n\} \in \langle E\rangle$ with $|A| = n + 1$, there exists a continuous function $\varphi_A : \Delta_n \to \Gamma(A)$ such that $\varphi_A(\Delta_J) \subseteq \Gamma(B)$ for any $B \in \langle A\rangle$ with $|B| = |J|$, where $J \subseteq \{0, \cdots, n\}$ and Δ_J denotes the face of Δ_n corresponding to B.*

In this paper, we assume that a G-convex space $(E;\Gamma)$ always satisfies the extra condition: $x \in \Gamma(\{x\})$ for any $x \in E$, and the Γ is also denoted by Γ^E if it is necessary.
A subset K of a G-convex space $(E;\Gamma)$ is said to be Γ-convex if for any $A \in \langle K\rangle$, $\Gamma(A) \subseteq K$. For a nonempty subset Q of E, the Γ-convex hull of Q, denoted by $\Gamma\text{-co}(Q)$, is defined by

$$\Gamma\text{-co}(Q) = \cap\{C : Q \subseteq C \subseteq E, C \text{ is } \Gamma\text{-convex}\}.$$

It is easy to see that $\Gamma\text{-co}(Q)$ is the smallest Γ-convex subset of E containing Q.

Definition 1.5. *A G-convex uniform space $(E;\mathcal{U},\Gamma)$ is a G-convex space so that its topology is induced by a uniformity $\mathcal{U}$. A G-convex uniform space $(E;\mathcal{U},\Gamma)$ is said to be a locally G-convex uniform space if the uniformity $\mathcal{U}$ has a base $\mathcal{B}$ consisting of open symmetric entourages such that for each $V \in \mathcal{B}$ and any $x \in E$, $V[x] := \{y \in X : (x, y) \in V\}$ is Γ-convex.*

For details of uniform spaces we refer to Kelley [6].

Tan and Zhang [11] showed that the product of an arbitrary family of G-convex spaces is a G-convex space: *Suppose $\{(E_i;\Gamma_i)\}_{i\in I}$ is any family of G-convex spaces. Let $E = \Pi_{i\in I}E_i$ be equipped with product topology. For each $i \in I$, let $\pi_i : E \to E_i$ be the projection. Define $\Gamma = \Pi_{i\in I}\Gamma_i : \langle E\rangle \multimap E$ by*

$$\Gamma(A) = \Pi_{i\in I}\Gamma_i(\pi_i(A))$$

for each $A \in \langle E\rangle$. Then $(E;\Gamma)$ is a G-convex space.

The following Lemmas 1.6, 1.7 and 1.8 which will be quoted in section 2 can be found in [4] and [5].

Lemma 1.6. *Let $(E;\Gamma) = (\Pi_{i\in I}E_i;\Pi_{i\in I}\Gamma_i)$ be the product G-convex space of a family of G-convex spaces $(E_i;\Gamma_i)$, $i \in I$. Then $K = \Pi_{i\in I}K_i$ is Γ-convex in E provided for each $i \in I$, K_i is a Γ_i-convex subset of E_i.*

Lemma 1.7. *Suppose $\{(E_i;\mathcal{U}_i,\Gamma_i)\}_{i\in I}$ is any family of locally G-convex uniform spaces. Let $E = \Pi_{i\in I}E_i$ be equipped with product topology and $\mathcal{U} = \Pi_{i\in I}\mathcal{U}_i$*

be the product uniformity on E. Then $(E;\mathcal{U},\Gamma)$ is a locally G-convex uniform space.

Lemma 1.8. *If K is a Γ-convex subset of a locally G-convex uniform space $(E;\mathcal{U},\Gamma)$, then its closure $\mathrm{cl}(K)$ is also Γ-convex.*

2. Coincidence and Fixed Point Theorems

Throughout this paper, $(E;\Gamma) = (\Pi_{i\in I}E_i;\Pi_{i\in I}\Gamma_i)$ denotes the product G-convex space of a family of G-convex spaces $(E_i;\Gamma_i)$, $i \in I$, and for each $i \in I$, let $E^i = \Pi_{j\in I\setminus\{i\}}E_j$ be equipped with the product convex structure $\Gamma^i = \Pi_{j\in I\setminus\{i\}}\Gamma_j$. Any point $x \in E$ is expressed as $x = [x^i, x_i]$, where $x^i \in E^i$ and $x_i \in E_i$.

Following Park [9], we make the following definition.

Definition 2.1. *Let X be a topological space and $(E;\Gamma)$ a G-convex space. A multimap $T : X \to 2^E$ is called a generalized Φ-mapping if there exists a multimap $S : X \to 2^E$ satisfying that*
(2.1.1) *for any $x \in X$ and any $A \in \langle S(x)\rangle$, $\Gamma\text{-co}(A) \subseteq T(x)$;*
(2.1.2) $X = \cup_{y\in E}\mathrm{cint}(S^-(y))$.
The mapping S is called a companion mapping of T.

We are now able to establish the following coincidence result which generalizes the Theorem 3.1 of Lin [8] from topological vector spaces to G-convex spaces.

Theorem 2.2. *Suppose for each $i \in I$, the two multimaps $F_i : E_i \to 2^{E^i}$ and $T_i : E^i \to 2^{E_i}$ satisfy the following conditions:*
(2.2.1) *F_i is a generalized Φ-mapping with the companion mapping H_i;*
(2.2.2) *T_i is a generalized Φ-mapping with the companion mapping S_i;*
(2.2.3) *There exists a nonempty compact subset M_i of E_i such that for each $Q^i \in \langle E^i\rangle$ there is a compact Γ^i-convex subset L_{Q^i} of E^i with*

$$Q^i \subseteq L_{Q^i}$$

and

$$E_i \setminus M_i \subseteq \cup_{x^i\in L_{Q^i}}\mathrm{cint}(H_i^-(x^i));$$

(2.2.4) *There exists a nonempty compact subset* K^i *of* E^i *such that for each* $N_i \in \langle E_i \rangle$ *there is a compact* Γ_i*-convex subset* L_{N_i} *of* E_i *with*

$$N_i \subseteq L_{N_i}$$

and

$$E^i \setminus K^i \subseteq \cup_{y_i \in L_{N_i}} \mathrm{cint}(S_i^-(y_i)).$$

Then there exist $\bar{x} = (\bar{x}_i)_{i \in I}$ *and* $\bar{y} = (\bar{y}_i)_{i \in I}$ *in* E *such that for each* $i \in I$, $\bar{y}_i \in T_i(\bar{x}^i)$ *and* $\bar{x}^i \in F_i(\bar{y}_i)$.

Proof. For each $i \in I$, we have by (2.2.1)

$$E_i = \cup_{x^i \in E^i} \mathrm{cint}(H_i^-(x^i)),$$

and then the compactness of M_i implies there is $Q^i \in \langle E^i \rangle$ such that

$$M_i \subseteq \cup_{x^i \in Q^i} \mathrm{cint}(H_i^-(x^i)). \tag{1}$$

In a like manner, there is $N_i \in \langle E_i \rangle$ such that

$$K^i \subseteq \cup_{y_i \in N_i} \mathrm{cint}(S_i^-(y_i)). \tag{2}$$

By (2.2.3), we have

$$L_{N_i} \setminus M_i \subseteq E_i \setminus M_i \subseteq \cup_{x^i \in L_{Q^i}} \mathrm{cint}(H_i^-(x^i)). \tag{3}$$

It follows from (1) and (3) that

$$L_{N_i} \subseteq \cup_{x^i \in L_{Q^i}} \mathrm{cint}(H_i^-(x^i)). \tag{4}$$

Similarly, (2) and (2.2.4) imply that

$$L_{Q^i} \subseteq \cup_{y_i \in L_{N_i}} \mathrm{cint}(S_i^-(y_i)). \tag{5}$$

Noting that both of L_{N_i} and L_{Q^i} are compact, it follows from (4) and (5) that there are $\{a_0^i, \cdots, a_{m_i}^i\} \in \langle L_{Q^i} \rangle$ and $\{b_{i,0}, \cdots, b_{i,l_i}\} \in \langle L_{N_i} \rangle$ such that

$$L_{N_i} \subseteq \cup_{j=0}^{m_i} \mathrm{cint}(H_i^-(a_j^i)) \tag{6}$$

and

$$L_{Q^i} \subseteq \cup_{j=0}^{l_i} \mathrm{cint}(S_i^-(b_{i,j})) \tag{7}$$

Let $A^i = \mathrm{cl}(\Gamma^i(\{a_0^i, \cdots, a_{m_i}^i\}))$ and $B_i = \mathrm{cl}(\Gamma_i(\{b_{i,0}, \cdots, b_{i,l_i}\}))$. B_i and A^i are closed subsets of L_{N_i} and L_{Q^i} respectively, and we have from (6) and (7) that

$$B_i = \cup_{j=0}^{m_i} \left(\mathrm{cint}(H_i^-(a_j^i)) \cap B_i\right) \tag{8}$$
$$A^i = \cup_{j=0}^{l_i} \left(\mathrm{cint}(S_i^-(b_{i,j})) \cap A^i\right). \tag{9}$$

By the definition of a G-convex space, there exists a continuous mapping $\varphi^i : \Delta_{m_i} \to A^i$ such that

$$\varphi^i(\Delta_J) \subseteq \Gamma^i(C) \subseteq A^i \tag{10}$$

for any $C \in \left\langle\{a_0^i, \cdots, a_{m_i}^i\}\right\rangle$ with $|C| = |J|$, where $J \subseteq \{0, \cdots, m_i\}$. Meanwhile, since B_i is compact, (8) shows that there is a partition of unity $\{\lambda_{i,j}\}_{j=0}^{m_i}$ subordinated to $\left\{\mathrm{cint}\left(H_i^-(a_j^i)\right) \cap B_i\right\}_{j=0}^{m_i}$. For any $j \in \{0, \cdots, m_i\}$ and any $x_i \in B_i$, we have

$$\begin{aligned} \lambda_{i,j}(x_i) \neq 0 &\Leftrightarrow x_i \in \mathrm{cint}\left(H_i^-(a_j^i)\right) \cap B_i \\ &\Rightarrow a_j^i \in H_i(x_i). \end{aligned} \tag{11}$$

Define a mapping $\psi_i : B_i \to \Delta_{m_i}$ by

$$\psi_i(x_i) = \sum_{j=0}^{m_i} \lambda_{i,j}(x_i)\mathbf{e}_j.$$

Clearly, ψ_i is continuous, and for any $x_i \in B_i$, one has

$$\psi_i(x_i) = \sum_{j \in J(x_i)} \lambda_{i,j}(x_i)\mathbf{e}_j \in \Delta_{J(x_i)},$$

where $J(x_i) = \{j \in \{0, \cdots, m_i\} : \lambda_{i,j}(x_i) \neq 0\}$. We see from (11) that $\{a_j^i : j \in J(x_i)\} \in \langle H_i(x_i)\rangle$, so in view of F_i being a generalized Φ-mapping with the companion mapping H_i, we conclude that

$$\Gamma^i\left(\{a_j^i : j \in J(x_i)\}\right) \subseteq \Gamma^i\text{-co}\left(\{a_j^i : j \in J(x_i)\}\right) \subseteq F_i(x_i).$$

Consequently, the function $f_i : B_i \to A^i$ defined by $f_i(x_i) = \varphi^i \circ \psi_i(x_i)$ has the property that

$$\begin{aligned} f_i(x_i) &= \varphi^i\left(\psi_i(x_i)\right) \\ &\in \varphi^i\left(\Delta_{J(x_i)}\right) \\ &\subseteq \Gamma^i(C), \quad \text{where } C = \{a_j^i : j \in J(x_i)\} \\ &\subseteq F_i(x_i), \end{aligned}$$

that is, f_i is a continuous selection of $F_i|_{B_i}$.

Reasoning as in the last paragraph, there are continuous functions $\varphi_i : \Delta_{l_i} \to B_i$ and $\psi^i : A^i \to \Delta_{l_i}$ such that their composition $g_i = \varphi_i \circ \psi^i : A^i \to B_i$ is a continuous selection of $T_i|_{A^i}$, that is,

$$g_i(x^i) \in T_i(x^i), \quad \forall x^i \in A^i.$$

let $\Delta = \Pi_{i\in I}\Delta_{l_i}$, which is a compact convex subset of the locally convex topological space $\Pi_{i\in I}\mathbb{R}^{l_i+1}$. Define a continuous mapping $\Omega : \Delta \to \Delta$ by

$$\Omega(t) = \Pi_{i\in I}\left(\psi^i \circ \varphi^i \circ \psi_i \circ \varphi_i \circ \pi_i\right)(t)$$

where π_i is the projection of Δ onto Δ_{l_i}. By Tychonoff's fixed point theorem, there is $\bar{t} = (\bar{t}_i)_{i\in I}$ in Δ such that

$$\begin{aligned}\bar{t} &= \Omega(\bar{t}) \\ &= \Pi_{i\in I}\psi^i \circ (\varphi^i \circ \psi_i)(\varphi_i(\bar{t}_i)).\end{aligned}$$

For each $i \in I$, let $\bar{y}_i = \varphi_i(\bar{t}_i)$, $\bar{x}^i = f_i(\bar{y}_i)$, and put $\bar{x} = (\bar{x}_i)_{i\in I}, \bar{y} = (\bar{y}_i)_{i\in I}$. Then from $\bar{t} = (\bar{t}_i)_{i\in I} = \Pi_{i\in I}\psi^i\left(f_i(\bar{y}_i)\right) = \Pi_{i\in I}\psi^i(\bar{x}^i)$ we obtain that

$$\bar{t}_i = \psi^i(\bar{x}^i)$$

and

$$\begin{aligned}\bar{y}_i &= \varphi_i(\bar{t}_i) = \varphi_i\left(\psi^i(\bar{x}^i)\right) = g_i(\bar{x}^i) \in T_i(\bar{x}^i) \\ \bar{x}^i &= f_i(\bar{y}_i) \in F_i(\bar{y}_i)\end{aligned}$$

for any $i \in I$. This completes the proof. □

Theorem 2.3. *For any $i \in I$, let $(E_i;\mathcal{U}_i, \Gamma_i)$ be a locally G-convex uniform space so that the convex structure Γ_i has the property that Γ_i-co(C_i) is compact whenever C_i is a compact subset of E_i, and suppose $F_i : E_i \to 2^{E^i}$ and $T_i : E^i \to 2^{E_i}$ satisfy the following conditions:*
(2.3.1) *Both of F_i and T_i are generalized Φ-mappings with the companion mappings H_i and S_i respectively.*
(2.3.2) *there exist a nonempty compact subset M_i of E_i and a $Q^i \in \langle E^i\rangle$ such that*

$$E_i \setminus M_i \subseteq \cup_{x^i\in Q^i}\text{cint}\left(H_i^-(x^i)\right).$$

(2.3.3) *there exist a nonempty compact subset* K^i *of* E^i *and an* $N_i \in \langle E_i \rangle$ *such that*

$$E^i \setminus K^i \subseteq \cup_{y_i \in N_i} \mathrm{cint}\left(S_i^-(y_i)\right).$$

Then there exist $\bar{x} = (\bar{x}_i)_{i \in I}$ *and* $\bar{y} = (\bar{y}_i)_{i \in I}$ *in* E *such that for each* $i \in I$, $\bar{y}_i \in T_i(\bar{x}^i)$ *and* $\bar{x}^i \in F_i(\bar{y}_i)$.

Proof. For each $i \in I$, let

$$\begin{aligned} L_{N_i} &= \Gamma_i\text{-co}\,(M_i \cup N_i) \\ L_{Q^i} &= \mathrm{cl}\left(\Gamma^i\text{-co}(K^i \cup Q^i)\right). \end{aligned}$$

By the assumption on the convex structure Γ_i, L_{N_i} is a compact Γ_i-convex subset of E_i containing N_i so that

$$E^i \setminus K^i \subseteq \cup_{y_i \in L_{N_i}} \mathrm{cint}\left(S_i^-(y_i)\right).$$

And it follows from Lemmas 1.6, 1.7 and 1.8 that L_{Q^i} is a compact Γ^i-convex subset of E^i containing Q^i so that

$$E_i \setminus M_i \subseteq \cup_{x^i \in L_{Q^i}} \mathrm{cint}\left(H_i^-(x^i)\right).$$

Then, following the same argument as in the proof of the above theorem, the conclusion follows. □

Essentially the same way of thinking as in the proof of Theorem 2.2, we can prove the following fixed point theorem.

Theorem 2.4. *Suppose for each* $i \in I$, $T_i : E \to 2^{E_i}$ *satisfies the following conditions:*
(2.4.1) T_i *is a generalized* Φ*-mapping with the companion mapping* S_i.
(2.4.2) *There exists a nonempty compact subset* K_i *of* E_i *such that for any* $N_i \in \langle E_i \rangle$ *there is a compact* Γ_i*-convex subset* L_{N_i} *of* E_i *with*

$$N_i \subseteq L_{N_i}$$

and

$$E \setminus K \subseteq \cup_{y_i \in L_{N_i}} \mathrm{cint}\left(S_i^-(y_i)\right),$$

where $K = \Pi_{i\in I} K_i$.
Then there exists an $\bar{x} = (\bar{x}_i)_{i\in I} \in E$ *such that* $\bar{x}_i \in T_i(\bar{x})$ *for each* $i \in I$.

Proof. Since K is a compact subset of E and since $E = \cup_{y_i\in E_i} \text{cint}\left(S_i^-(y_i)\right)$, there is an $N_i \in \langle E_i \rangle$ such that

$$K \subseteq \cup_{y_i\in N_i} \text{cint}\left(S_i^-(y_i)\right). \tag{12}$$

By (2.4.2) there is a compact Γ_i-convex subset L_{N_i} of E_i such that $N_i \subseteq L_{N_i}$ and

$$E \setminus K \subseteq \cup_{y_i\in L_{N_i}} \text{cint}\left(S_i^-(y_i)\right). \tag{13}$$

Let $N = \Pi_{i\in I} N_i$ and $L_N = \Pi_{i\in I} L_{N_i}$. L_N is a compact Γ-convex subset of E containing N. Noting that $L_N \setminus K \subseteq E \setminus K$, it follows from (13) that

$$L_N \setminus K \subseteq \cup_{y_i\in L_{N_i}} \text{cint}\left(S_i^-(y_i)\right). \tag{14}$$

Since $N_i \subseteq L_{N_i}$ we have from (12) and (14) that $L_N \subseteq \cup_{y_i\in L_{N_i}} \text{cint}\left(S_i^-(y_i)\right)$, and hence the compactness of L_N implies that there is $A_i = \{y_{i,0}, \cdots, y_{i,l_i}\} \in \langle L_{N_i} \rangle$ with

$$L_N = \cup_{j=0}^{l_i} \left(\text{cint}\left(S_i^-(y_{i,j})\right) \cap L_N\right). \tag{15}$$

By the definition of a G-convex space, there exists a continuous mapping $\varphi_i : \Delta_{l_i} \to \Gamma(A_i)$ such that

$$\varphi_i\left(\Delta_J\right) \subseteq \Gamma(C) \subseteq \Gamma(A_i) \subseteq L_{N_i} \tag{16}$$

for any $C \in \langle A_i \rangle$ with $|C| = |J|$, where $J \subseteq \{0, \cdots, l_i\}$. Besides, (15) shows that there is a partition of unity $\{\lambda_{i,j}\}_{j=0}^{l_i}$ subordinated to $\left\{\text{cint}\left(S_i^-(y_{i,j})\right) \cap L_N\right\}_{j=0}^{l_i}$. For any $j \in \{0, \cdots, l_i\}$ and any $x \in L_N$, we have $y_{i,j} \in S_i(x)$ provided that $\lambda_{i,j}(x) \neq 0$. Define a mapping $\psi_i : L_N \to \Delta_{l_i}$ by

$$\psi_i(x) = \sum_{j=0}^{l_i} \lambda_{i,j}(x)\mathbf{e}_j.$$

Each ψ_i is continuous, and for any $x \in L_N$, one has

$$\psi_i(x) = \sum_{j\in J(x)} \lambda_{i,j}(x)\mathbf{e}_j \in \Delta_{J(x)},$$

where $J(x) = \{j \in \{0, \cdots, l_i\} : \lambda_{i,j}(x) \neq 0\}$. Noting that $\{y_{i,j} : j \in J(x)\} \in \langle S_i(x)\rangle$ and T_i is a generalized Φ-mapping with the companion mapping S_i, we see from (16) that the function $f_i : L_N \to L_{N_i}$ defined by $f_i(x) = \varphi_i \circ \psi_i(x)$ has the property that for any $x \in L_N$,

$$\begin{aligned} f_i(x) &= \varphi_i \circ \psi_i(x) \\ &\in \varphi_i\left(\Delta_{J(x)}\right) \\ &\subseteq \Gamma(C), \quad \text{where } C = \{y_{i,j} : j \in J(x)\} \\ &\subseteq T_i(x). \end{aligned}$$

Let $\Delta = \Pi_{i\in I}\Delta_{l_i}$. Define continuous mappings $\Omega : \Delta \to L_N$ and $\Psi : L_N \to \Delta$ by $\Omega(t) = \Pi_{i\in I}\varphi_i(\pi_i(t))$ for $t \in \Delta$, and $\Psi(x) = \Pi_{k\in I}\psi_k(x)$ for $x \in L_N$, where $\pi_i : \Delta \to \Delta_{l_i}$ is the projection of Δ onto Δ_{l_i}. By Tychonoff's fixed point theorem, the continuous function $\Psi \circ \Omega : \Delta \to \Delta$ has a fixed point $\bar{t}$. Let $\bar{x} = \Omega(\bar{t})$. Then,

$$\begin{aligned} \bar{x} &= \Omega(\bar{t}) \\ &= \Omega\left(\Psi \circ \Omega(\bar{t})\right) \\ &= (\Omega \circ \Psi)\left(\Omega(\bar{t})\right) \\ &= (\Omega \circ \Psi)(\bar{x}) \\ &= \Omega\left(\Pi_{k\in I}\psi_k(\bar{x})\right) \\ &= \Pi_{i\in I}\varphi_i\left(\pi_i\left(\Pi_{k\in I}\psi_k(\bar{x})\right)\right) \\ &= \Pi_{i\in I}\varphi_i(\psi_i(\bar{x})) \\ &= \Pi_{i\in I}(\varphi_i \circ \psi_i)(\bar{x}). \end{aligned}$$

Thus, $\bar{x}_i = \varphi_i \circ \psi_i(\bar{x}) = f_i(\bar{x}) \in T_i(\bar{x})$ for each $i \in I$. □

Just as Theorem 2.3, the following fixed point result holds.

Theorem 2.5. *For any $i \in I$, let $(E_i; \mathcal{U}_i, \Gamma_i)$ be a locally G-convex uniform space so that the convex structure Γ_i has the property that Γ_i-co(C_i) is compact whenever C_i is a compact subset of E_i, and suppose $T_i : E \to 2^{E_i}$ satisfy the following conditions:*

(2.5.1) *T_i is a generalized Φ-mapping with the companion mapping S_i.*

(2.5.2) *There exist a nonempty compact subset K_i of E_i and an $N_i \in \langle E_i\rangle$ such that*

$$E \setminus K \subseteq \cup_{y_i\in N_i}\text{cint}\left(S_i^-(y_i)\right),$$

where $K = \Pi_{i\in I} K_i$.
Then there exists an $\bar{x} = (\bar{x}_i)_{i\in I} \in E$ *such that* $\bar{x}_i \in T_i(\bar{x})$ *for each* $i \in I$.

3. System of Inequalities and Minimax Theorems

In this section, we give some immediate applications of our coincidence results to the existence of solution for a system of inequalities.

Definition 3.1. *Let* X *be a nonempty set,* $(Y;\Gamma)$ *a* G*-convex space, and* $f, g : X \times Y \to \mathbb{R}$. *Then*
(3.1.1) g *is said to be* f*-quasiconcave on* Y *if for any* $x \in X$ *and any* $A \in \langle Y\rangle$,

$$\min_{y\in A} f(x,y) \le g(x,z),\ \ \forall z \in \Gamma\text{-co}(A).$$

(3.1.2) g *is said to be* f*-quasiconvex on* Y *if for any* $x \in X$ *and any* $A \in \langle Y\rangle$,

$$\max_{y\in A} f(x,y) \ge g(x,z),\ \ \forall z \in \Gamma\text{-co}(A).$$

Theorem 3.2. *For any* $i \in I$, *let* $f_i, g_i, p_i, q_i : E^i \times E_i \to \mathbb{R}$ *be functions and* $\{a_i\}_{i\in I}$, $\{b_i\}_{i\in I}$ *be two families of real numbers. Suppose for each* $i \in I$, *the following conditions hold:*
(3.2.1) *For each* $x^i \in E^i$, $x_i \mapsto g_i(x^i, x_i)$ *is* f_i*-quasiconcave on* E_i, *and for each* $x_i \in E_i$, $x^i \mapsto q_i(x^i, x_i)$ *is* p_i*-quasiconvex on* E^i.
(3.2.2) *For each* $x_i \in E_i$, $x^i \mapsto f_i(x^i, x_i)$ *is transfer compactly l.s.c. on* E^i, *and for each* $x^i \in E^i$, $x_i \mapsto p_i(x^i, x_i)$ *is transfer compactly u.s.c. on* E_i.
(3.2.3) *For each* $x^i \in E^i$, *there is an* $x_i \in E_i$ *such that* $f_i(x^i, x_i) > a_i$.
(3.2.4) *For each* $x_i \in E_i$, *there is an* $x^i \in E^i$ *such that* $p_i(x^i, x_i) < b_i$.
(3.2.5) *There exists a nonempty compact subset* M_i *of* E_i *such that for each* $Q^i \in \langle E^i\rangle$ *there is a compact* Γ^i*-convex subset* L_{Q^i} *of* E^i *so that* $Q^i \subseteq L_{Q^i}$ *and for each* $y_i \in E_i \setminus M_i$ *there exists* $x^i \in L_{Q^i}$ *with*

$$y_i \in \text{cint}\left(\{v_i \in E_i : p_i(x^i, v_i) < b_i\}\right).$$

(3.2.6) *There exists a nonempty compact subset* K^i *of* E^i *such that for each* $N_i \in \langle E_i\rangle$ *there is a compact* Γ_i*-convex subset* L_{N_i} *of* E_i *so that* $N_i \subseteq L_{N_i}$ *and for each* $x^i \in E^i \setminus K^i$ *there exists* $y_i \in L_{N_i}$ *with*

$$x^i \in \text{cint}\left(\{u^i \in E^i : f_i(u^i, y_i) > a_i\}\right).$$

Then there are $\bar{x} = (\bar{x}_i)_{i\in I}$ *and* $\bar{y} = (\bar{y}_i)_{i\in I}$ *in* E *such that* $g_i(\bar{x}^i, \bar{y}_i) > a_i$ *and* $q_i(\bar{x}^i, \bar{y}_i) < b_i$ *for all* $i \in I$.

Proof. For each $i \in I$, define $S_i, T_i : E^i \to 2^{E_i}$ and $H_i, F_i : E_i \to 2^{E^i}$ by

$$\begin{aligned} S_i(x^i) &= \{y_i \in E_i : f_i(x^i, y_i) > a_i\} \\ T_i(x^i) &= \{y_i \in E_i : g_i(x^i, y_i) > a_i\} \\ H_i(x_i) &= \{u^i \in E^i : p_i(u^i, x_i) < b_i\} \\ F_i(x_i) &= \{u^i \in E^i : q_i(u^i, x_i) < b_i\}. \end{aligned}$$

Firstly, we show that T_i is a generalized Φ-mapping with the companion mapping S_i. To see this we have to show that (i) Γ_i-co$(A) \subseteq T_i(x^i)$ for any $x^i \in E^i$ and any $A \in \langle S_i(x^i)\rangle$, and (ii) $E^i = \cup_{y_i\in E_i}\text{cint}\left(S_i^-(y_i)\right)$. For any $x^i \in E^i$ and any $A \in \langle S_i(x^i)\rangle$, since $x_i \mapsto g_i(x^i, x_i)$ is f_i-quasiconcave, we have

$$a_i < \min_{y\in A} f_i(x^i, y) \leq g_i(x^i, z), \ \ \forall z \in \Gamma_i\text{-co}(A).$$

This completes the proof of (i). Next, the map $S_i^- : E_i \to 2^{E^i}$ defined by

$$S_i^-(x_i) = \left\{x^i \in E^i : f_i(x^i, x_i) > a_i\right\}$$

is transfer compactly open valued by (3.2.2) and Lemma 1.2, which in conjunction with (3.2.3) and Lemma 1.3 shows that (ii) holds. Similarly, F_i is a generalized Φ-mapping with the companion mapping H_i. Moreover, conditions (3.2.5) and (3.2.6) imply conditions (2.2.3) and (2.2.4) respectively. Therefore, all the requirements of Theorem 2.2 are satisfied, and thus the conclusion follows. □

Theorem 3.3. *For each* $i \in I$, *suppose* $f_i, g_i, p_i, q_i : E^i \times E_i \to \mathbb{R}$ *are functions satisfying the following conditions:*
(3.3.1) $f_i(x) \leq g_i(x) \leq p_i(x) \leq q_i(x)$ *for any* $x \in E$.
(3.3.2) *For each* $x^i \in E^i$, $x_i \mapsto g_i(x^i, x_i)$ *is* f_i*-quasiconcave on* E_i, *and for each* $x_i \in E_i$, $x^i \mapsto q_i(x^i, x_i)$ *is* p_i*-quasiconvex on* E^i.
(3.3.3) *For each* $x_i \in E_i$, $x^i \mapsto f_i(x^i, x_i)$ *is transfer compactly l.s.c. on* E^i, *and for each* $x^i \in E^i$, $x_i \mapsto p_i(x^i, x_i)$ *is transfer compactly u.s.c. on* E_i.
(3.3.4) *There exists a nonempty compact subset* M_i *of* E_i *such that for each* $Q^i \in \langle E^i\rangle$ *there is a compact* Γ^i*-convex subset* L_{Q^i} *of* E^i *so that* $Q^i \subseteq L_{Q^i}$ *and*

for each $y_i \in E_i \setminus M_i$ there exists $x^i \in L_{Q^i}$ with

$$y_i \in \operatorname{cint}\left(\{v_i \in E_i : p_i(x^i, v_i) \leq \sup_{v_i \in E_i} \inf_{u^i \in E^i} p_i(u^i, v_i)\}\right).$$

(3.3.5) *There exists a nonempty compact subset K^i of E^i such that for each $N_i \in \langle E_i \rangle$ there is a compact Γ_i-convex subset L_{N_i} of E_i so that $N_i \subseteq L_{N_i}$ and for each $x^i \in E^i \setminus K^i$ there exists $y_i \in L_{N_i}$ with*

$$x^i \in \operatorname{cint}\left(\{u^i \in E^i : f_i(u^i, y_i) \geq \inf_{u^i \in E^i} \sup_{v_i \in E_i} f_i(u^i, v_i)\}\right).$$

Then

$$\inf_{u^i \in E^i} \sup_{v_i \in E_i} f_i(u^i, v_i) \leq \sup_{v_i \in E_i} \inf_{u^i \in E^i} q_i(u^i, v_i)$$

for any $i \in I$.

Proof. Without loss of generality, we may assume that for any $i \in I$,

$$\inf_{u^i \in E^i} \sup_{v_i \in E_i} f_i(u^i, v_i) > -\infty \quad \text{and} \quad \sup_{v_i \in E_i} \inf_{u^i \in E^i} q_i(u^i, v_i) < \infty.$$

For any $i \in I$, let a_i and b_i be any real numbers satisfying that

$$a_i < \inf_{u^i \in E^i} \sup_{v_i \in E_i} f_i(u^i, v_i) \quad \text{and} \quad b_i > \sup_{v_i \in E_i} \inf_{u^i \in E^i} q_i(u^i, v_i) \geq \sup_{v_i \in E_i} \inf_{u^i \in E^i} p_i(u^i, v_i).$$

Then for each $x^i \in E^i$, there exists an $x_i \in E_i$ such that $f_i(x^i, x_i) > a_i$, and for any $x_i \in E_i$, there exists an $x^i \in E^i$ such that $p_i(x^i, x_i) < b_i$. So conditions (3.2.3) and (3.2.4) are satisfied. Moreover, conditions (3.3.4) and (3.3.5) imply conditions (3.2.5) and (3.2.6) respectively. Consequently, all the requirements of Theorem 3.2 are satisfied, and so there exist $\bar{x} = (\bar{x}_i)_{i \in I}$ and $\bar{y} = (\bar{y}_i)_{i \in I}$ in E such that $g_i(\bar{x}^i, \bar{y}_i) > a_i$ and $q_i(\bar{x}^i, \bar{y}_i) < b_i$ for all $i \in I$. From $q_i(\bar{x}^i, \bar{y}_i) \geq g_i(\bar{x}^i, \bar{y}_i)$, we obtain that $b_i > a_i$. Since both of a_i and b_i with $a_i < \inf_{u^i \in E^i} \sup_{v_i \in E_i} f_i(u^i, v_i)$ and $b_i > \sup_{v_i \in E_i} \inf_{u^i \in E^i} q_i(u^i, v_i)$ are arbitrary, we conclude that

$$\inf_{u^i \in E^i} \sup_{v_i \in E_i} f_i(u^i, v_i) \leq \sup_{v_i \in E_i} \inf_{u^i \in E^i} q_i(u^i, v_i)$$

□

Corollary 3.4. *In Theorem 3.3, if $f_i = g_i = p_i = q_i$ for all $i \in I$, then*

$$\sup_{v_i \in E_i} \inf_{u^i \in E^i} f_i(u^i, v_i) = \inf_{u^i \in E^i} \sup_{v_i \in E_i} f_i(u^i, v_i).$$

To end this section, we like to remark that in the seting of locally G-convex uniform spaces if for each $i \in I$, the convex structure Γ_i has the property that Γ_i-co(C_i) is compact whenever C_i is a compact subset of E_i, then conditions (3.2.5), (3.2.6), (3.3.4) and (3.3.5) can be replaced with the following conditions (3.2.5)′, (3.2.6)′, (3.3.4)′ and (3.3.5)′ respectively.

(3.2.5)′ *There exist a nonempty compact subset M_i of E_i and an $Q^i \in \langle E^i \rangle$ such that for each $y_i \in E_i \setminus M_i$ there is $x^i \in Q^i$ with*

$$y_i \in \text{cint}\left(\{v_i \in E_i : p_i(x^i, v_i) < b_i\}\right).$$

(3.2.6)′ *There exist a nonempty compact subset K^i of E^i and an $N_i \in \langle E_i \rangle$ such that for each $x^i \in E^i \setminus K^i$ there is $y_i \in N_i$ with*

$$x^i \in \text{cint}\left(\{u^i \in E^i : f_i(u^i, y_i) > a_i\}\right).$$

(3.3.4)′ *There exist a nonempty compact subset M_i of E_i and an $Q^i \in \langle E^i \rangle$ such that*

$$E_i \setminus M_i \subseteq \cup_{x^i \in Q^i} \text{cint}\left(\{v_i \in E_i : p_i(x^i, v_i) \leq \sup_{v_i \in E_i} \inf_{u^i \in E^i} p_i(u^i, v_i)\}\right).$$

(3.3.5)′ *There exist a nonempty compact subset K^i of E^i and an $N_i \in \langle E_i \rangle$ such that*

$$E^i \setminus K^i \subseteq \cup_{y_i \in N_i} \text{cint}\left(\{u^i \in E^i : f_i(u^i, y_i) \geq \inf_{u^i \in E^i} \sup_{v_i \in E_i} f_i(u^i, v_i)\}\right).$$

4. Maximal Elements and Equilibrium Points

In this section we shall apply our fixed point results in section 2 to obtain some theorems about maximal elements, nonempty intersection, and equilibrium points for an abstract economy.

An abstract economy is a family $\mathcal{E} = ((E_i; \Gamma_i), A_i, B_i, P_i)_{i \in I}$, where I is a finite or infinite set of agents and for each $i \in I$, each commodity space $(E_i; \Gamma_i)$ is a G-convex space, $A_i, B_i : E = \Pi_{i \in I} E_i \to 2^{E_i}$ are the constraint correspondences, and $P_i : E \to 2^{E_i}$ is the preference correspondence. A point $\bar{x} \in E$ is said to be an equilibrium point of an abstract economy $\mathcal{E} = ((E_i;\Gamma_i), A_i, B_i, P_i)_{i\in I}$ if $\bar{x}_i \in B_i(\bar{x})$ and $A_i(\bar{x}) \cap P_i(\bar{x}) = \varnothing$ for each $i \in I$.

Theorem 4.1. *Let $\mathcal{E} = ((E_i;\Gamma_i), A_i, B_i, P_i)_{i\in I}$ be an abstract economy. Suppose the following conditions hold:*
(4.1.1) *For each $x \in E$, $A_i(x) \neq \varnothing$, and for each $N \in \langle A_i(x)\rangle$, Γ_i-co$(N) \subseteq B_i(x)$.*
(4.1.2) *For each $x = (x_i)_{i\in I} \in E$, $x_i \notin \Gamma_i$-co$(P_i(x))$ for every $i \in I$.*
(4.1.3) *For each $y_i \in E_i$, all of $A_i^-(y_i)$, $P_i^-(y_i)$ and $C_i = \{x \in E : P_i(x) \cap A_i(x) = \varnothing\}$ are compactly open such that $E = \cup_{y_i \in E_i}\left[\left(P_i^-(y_i) \cup C_i\right) \cap A_i^-(y_i)\right]$.*
(4.1.4) *There exists a nonempty compact subset K_i of E_i such that for any $N_i \in \langle E_i\rangle$ there is compact Γ_i-convex subset L_{N_i} of E_i with $N_i \subseteq L_{N_i}$ and*

$$E \setminus K \subseteq \cup_{y_i \in L_{N_i}}\left[\left(P_i^-(y_i) \cup C_i\right) \cap A_i^-(y_i)\right],$$

where $K = \Pi_{i\in I}K_i$.
Then $\mathcal{E}$ has an equilibrium point.

Proof. For each $i \in I$, let $D_i = E \setminus C_i$ and define $S_i, T_i : E \to 2^{E_i}$ by

$$S_i(x) = \begin{cases} P_i(x) \cap A_i(x), & \text{if } x \in D_i; \\ A_i(x), & \text{if } x \in C_i, \end{cases}$$

and

$$T_i(x) = \begin{cases} \Gamma_i\text{-co}\,(P_i(x)) \cap B_i(x), & \text{if } x \in D_i; \\ B_i(x), & \text{if } x \in C_i. \end{cases}$$

For each $x \in E$ and each $N \in \langle S_i(x)\rangle$, it is easy to check that Γ_i-co$(N) \subseteq T_i(x)$. Also, for each $y_i \in E_i$, we see from

$$\begin{aligned} S_i^-(y_i) &= \left(P_i^-(y_i) \cap A_i^-(y_i) \cap D_i\right) \cup \left(A_i^-(y_i) \cap C_i\right) \\ &= \left(P_i^-(y_i) \cap A_i^-(y_i)\right) \cup \left(A_i^-(y_i) \cap C_i\right) \\ &= \left(P_i^-(y_i) \cup C_i\right) \cap A_i^-(y_i) \end{aligned}$$

and (4.1.3) that that $S_i^-(y_i)$ is compactly open and $E = \cup_{y_i \in E_i} S_i^-(y_i)$. Thus we have shown that T_i is generalized Φ-mapping with the companion mapping S_i, which together with (4.1.4) shows that all the requirements of Theorem 2.4 are satisfied. Thus, there is an $\bar{x} \in E$ such that $\bar{x}_i \in T_i(\bar{x})$ for all $i \in I$. Now, (4.1.2) implies that $\bar{x} \in C_i$, hence we infer that $\bar{x}_i \in B_i(\bar{x})$ and $P_i(\bar{x}) \cap A_i(\bar{x}) = \varnothing$ for all $i \in I$. This completes the proof. $\square$

Another immediate consequence of Theorem 2.4 is the following existence theorem of maximal elements.

Theorem 4.2 *Let $(E;\Gamma)$ be a G-convex space and $S, T : E \to 2^E$ satisfy the following conditions:*
(4.2.1) *For each $x \in E$, $\Gamma\text{-co}(S(x)) \subseteq T(x)$.*
(4.2.2) *S^- is transfer compactly open valued.*
(4.2.3) *There exist a nonempty compact subset K and a nonempty compact Γ-convex subset C of E such that $E \setminus K \subseteq \cup_{y\in C}\text{cint}\,(S^-(y))$.*
(4.2.4) *T does not have a fixed point on E.*
Then S has a maximal element, that is, there is an $\bar{x} \in E$ such that $S(\bar{x}) = \varnothing$.

Proof. If $S(x) \neq \varnothing$ for any $x \in E$, then by Lemma 1.3

$$E = \cup_{y\in E}\text{cint}\left(S^-(y)\right),$$

which in combining with (4.2.1) and (4.2.2) gives us that T is a generalized Φ-mapping with the companion mapping S. Then in view of (4.2.3) we infer from Theorem 2.4 that T has a fixed point, which contradicts (4.2.4). Therefore S must have a maximal element. $\square$

Theorem 4.2 can be further extended as:

Theorem 4.3. *Suppose I is a finite index set, and for each $i \in I$, $S_i, T_i : E \to 2^{E_i}$ satisfy the following conditions:*
(4.3.1) *For all $x \in E$, $\Gamma_i\text{-co}\,(S_i(x)) \subseteq T_i(x)$.*
(4.3.2) *S_i^- is transfer compactly open valued.*
(4.3.3) *For all $x \in E$, $x_i \notin T_i(x)$.*
(4.3.4) *There exist a nonempty compact subset K of E and a nonempty compact Γ_i-convex subset C_i of E_i such that for all $x \in E \setminus K$ and for all $i \in I$ with $S_i(x) \neq \varnothing$ there is $\hat{y}_i \in C_i$ with $x \in \text{cint}\left(S_i^-(\hat{y}_i)\right)$.*
Then there exists $\bar{x} \in E$ such that $S_i(\bar{x}) = \varnothing$ for all $i \in I$.

Proof. For each $x \in E$, let $I(x) = \{i \in I : S_i(x) \neq \varnothing\}$. We have to prove that $I(\overline{x}) = \varnothing$ for some $\overline{x} \in E$. On the contrary, assume that $I(x) \neq \varnothing$ for all $x \in X$. For $i \in I$ and $x \in E$, let $A_i(x) = E^i \times S_i(x)$ and $B_i(x) = E^i \times T_i(x)$, and define $F, G : E \to 2^E$ by

$$F(x) = \cap_{i \in I(x)} A_i(x)$$

and

$$G(x) = \cap_{i \in I(x)} B_i(x).$$

It follows from (4.3.1) that for each $x \in E$,

$$\begin{aligned}\Gamma\text{-co}\,(F(x)) &= \Gamma\text{-co}\left[\cap_{i \in I(x)}\left(E^i \times S_i(x)\right)\right] \\ &\subseteq \cap_{i \in I(x)}\left(E^i \times \Gamma\text{-co}(S_i(x))\right) \\ &\subseteq \cap_{i \in I(x)}\left(E^i \times T_i(x)\right) = G(x),\end{aligned}$$

and (4.3.3) leads that $x \notin G(x)$. We now show that F^- is transfer compactly open valued. Let $y \in E$ and $x \in F^-(y)$. Noting that $y \in F(x) = \cap_{i \in I(x)} A_i(x) = \cap_{i \in I(x)}\left(E^i \times S_i(x)\right)$, we see that $y_i \in S_i(x)$ for all $i \in I(x)$, that is, $x \in S_i^-(y_i)$ for all $i \in I(x)$. Since S_i^- is transfer compactly open valued, there is $\hat{y}_i \in E_i$ such that

$$x \in \text{cint}\left(S_i^-(\hat{y}_i)\right) \tag{17}$$

For each $i \in I \setminus I(x)$ choose $z_i \in E_i$, and define $\tilde{y} \in E$ by

$$\tilde{y}_i = \begin{cases} \hat{y}_i, & \text{if } i \in I(x); \\ z_i, & \text{if } i \in I \setminus I(x). \end{cases} \tag{18}$$

Since

$$\begin{aligned} & z \in A_i^-(\tilde{y}) \\ \Leftrightarrow\ & \bar{y} \in A_i(z) = E^i \times S_i(x) \\ \Leftrightarrow\ & \hat{y}_i \in S_i(z) \\ \Leftrightarrow\ & z \in S_i^-(\hat{y}_i),\end{aligned}$$

we conclude that $A_i^-(\tilde{y}) = S_i^-(\hat{y}_i)$, and so by (17)

$$x \in \text{cint}\left(A_i^-(\tilde{y})\right)$$

for all $i \in I(x)$. Thus,

$$x \in \cap_{i\in I(x)} \text{cint}\left(A_i^-(\bar{y})\right) \subseteq \text{cint}\left(\cap_{i\in I(x)} A_i^-(\bar{y})\right) = \text{cint}\left(F^-(\bar{y})\right)$$

once we notice that $I(x)$ is a finite set. This shows that F^- is transfer compactly open valued. By (4.3.4), for all $x \in E\setminus K$ and for each $i \in I(x)$, there is a $\hat{y}_i \in C_i$ such that $x \in \text{cint}\left(S_i^-(\hat{y}_i)\right) = \text{cint}\left(A_i(\tilde{y})\right)$, where $\tilde{y}$ is defined as in (18). Let $C = \Pi_{i\in I} C_i$, which is a compact Γ-convex subset of E. Then

$$\begin{aligned} x &\in \cap_{i\in I(x)} \text{cint}\left(A_i^-(\tilde{y})\right) \\ &\subseteq \text{cint}\left(\cap_{i\in I(x)} A_i^-(\tilde{y})\right) = \text{cint}(F^-(\tilde{y})), \end{aligned}$$

that is, $E \setminus K \subseteq \cup_{y\in C}\text{cint}(F^-(\tilde{y}))$. By means of Theorem 4.2, there is $\hat{z} \in E$ such that $F(\hat{z}) = \varnothing$. Since $I(\hat{z}) \neq \varnothing$, $S_i(\hat{z}) \neq \varnothing$ and $A_i(\hat{z}) \neq \varnothing$ for all $i \in I(\hat{z})$. Hence, $F(\hat{z}) = \cap_{i\in I(\hat{z})} A_i(\hat{z}) \neq \varnothing$, which contradicts the fact that $F(\hat{z}) = \varnothing$. This completes the proof. □

Finally, we deduce a nonempty intersection theorem.

Theorem 4.4. *Suppose $\{A_i\}_{i\in I}$ and $\{B_i\}_{i\in I}$ are two families of nonempty subsets of the product G-convex space $E = \Pi_{i\in I} E_i$ satisfing that:*
(4.4.1) *For each $x^i \in E^i$ and each $N \in \langle A_i[x^i]\rangle$, Γ_i-co$(N) \subseteq B_i[x^i]$ and there exists a point $y_i \in E_i$ such that $x^i \in \text{cint}\left(A_i[y_i]\right)$, where*

$$\begin{aligned} A_i[x^i] &= \{y_i \in E_i : [x^i, y_i] \in A_i\} \\ B_i[x^i] &= \{y_i \in E_i : [x^i, y_i] \in B_i\} \\ A_i[y_i] &= \{x^i \in E^i : [x^i, y_i] \in A_i\}. \end{aligned}$$

(4.4.2) *There exists a nonempty compact subset K_i of E_i such that for any $N_i \in \langle E_i\rangle$ there is a compact Γ_i-convex subset L_{N_i} with $N_i \subseteq L_{N_i}$ and*

$$E \setminus K \subseteq \cup_{y_i\in L_{N_i}} \left(\text{cint}(A_i[y_i]) \times E_i\right),$$

where $K = \Pi_{i\in I} K_i$.
Then $\cap_{i\in I} B_i \neq \varnothing$.

Proof. For each $i \in I$, define $S_i, T_i : E \to 2^{E_i}$ by $S_i(x) = A_i[x^i]$ and $T_i(x) =$

$B_i[x^i]$ for all $x \in E$. It follows from (4.4.1) that Γ_i-co$(N) \subseteq T_i(x)$ for any $x \in E$ and any $N \in \langle S_i(x) \rangle$. Besides, for each $y_i \in E_i$, we have

$$\begin{aligned} S_i^-(y_i) &= \{x = [x^i, x_i] \in E : y_i \in S_i(x)\} \\ &= \{x = [x^i, x_i] \in E : y_i \in A_i[x^i]\} \\ &= \{x = [x^i, x_i] \in E : x^i \in A_i[y_i]\} \\ &= A_i[y_i] \times E_i \\ &\supseteq \operatorname{cint}(A_i[y_i]) \times E_i, \end{aligned}$$

hence $\operatorname{cint}\left(S_i^-(y_i)\right) \supseteq \operatorname{cint}(A_i[y_i]) \times E_i$ for each $y_i \in E_i$. By (4.4.1), for each $x \in E$, there is a $y_i \in E_i$ such that

$$x = [x^i, x_i] \in \operatorname{cint}(A_i[y_i]) \times E_i \subseteq \operatorname{cint}\left(S_i^-(y_i)\right).$$

This shows that $E = \cup_{y_i \in E_i} \operatorname{cint}\left(S_i^-(y_i)\right)$. And thus T_i is a generalized Φ-mapping with the companion mapping S_i, which together with (4.4.2) leads to the existence of a point $\bar{x}$ in E so that $\bar{x}_i \in T_i(\bar{x}) = B_i(\bar{x}^i)$ for each $i \in I$. In other words, $\bar{x} \in \cap_{i \in I} B_i$. □

References

1. Q.H. Ansari, A. Idzik and J.C. Yao, Coincidence and fixed point theorems with applications, Topol. Methods Nonlinear Anal. 15(2000)191-202.

2. X.P. Ding and J.Y. Park, Collectively fixed point theorem and abstract economy in G-convex spaces, Numer. Funct. Anal. Optimi. 23(2002)779-790.

3. X.P. Ding, New H-KKM theorems and their applications to geometric property, coincidence theorems, minimax inequality and maximal elements, Indian J. Pure Appl. Math. 26(1995)1-19.

4. Y.Y. Huang, T.Y. Kuo and J.C. Jeng, Fixed point theorems for condensing multimaps on locally G-convex spaces, Nonlinear Analysis, 67(2007)1522-1531.

5. T.Y. Kuo, J.C. Jeng and Y.Y. Huang, Fixed point theorems for compact-multimaps on almost Γ-convex sets in generalized convex spaces, Nonlinear Analysis, 66(2007)415-426.

6. J.L. Kelley, General Topology, Van Nostrand, Princeton, NJ, 1955.

7. L.J. Lin and Q.H. Ansari, Collective fixed points and maximal elements with applications to abstract economies, J. Math. Anal. Appl. 296(2004)455-472.

8. L.J. Lin, System of coincidence theorems with applications, J. Math. Anal Appl. 285(2003)408-418.

9. S. Park, Continuous selection theorems in generalized convex spaces, Numer. Funct. Anal. Optimi. 20(1999)567-583.

10. S. Park and H. Kim, Coincidence theorems for admissible multifunctions on generalized convex spaces. J. Math. Anal. Appl. 197(1996)173-187.

11. K.K. Tan and X.L. Zhang, Fixed point theorems on G-convex spaces and applications, in: Proceedings of Nonlinear Functional Analysis and Applications, Vol. 1, Kyungnam University, Masan, Korea, 1996, pp. 1-19.

12. E. Tarafdar, Fixed point theorems in H-spaces and equilibrium points of abstract economies, J. Austral. Math. Soc.(Series A) 53(1992)252-260.

13. G. Tian and J. Zhou, Transfer continuities, generalizations of the Weierstrass and maximum theorems: a full characterization, J. Math. Econom. 24(1995)281-303.